DEVELOPMENT AND ANALYSIS OF BULK QUEUEING MODELS WITH APPLICATIONS

THESIS

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By

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Certificate

This is to certify that Mr. Anis Ahmad has duly completed his thesis entitled, **"Development and analysis of bulk queueing models with applications"** for the degree of Ph.D. in mathematics from Bundelkhand University, Jhansi and his thesis is upto the mark under our supervision.

We further certify that this work has been originally done by him and he has put up the minimum required attendance of the department since the date of his registration.

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Ahmad

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PREFACE

The work presented in this thesis is the out come of research is carried out in the field of queueing theory, at the department of Bundelkhand University Jhansi, while engaged in teaching as a Lecturer, in the department of Mathematical science & computer Application, Bundelkhand University, Jhansi.

The work presented in the thesis is based on my following research papers.

1- The steady state model of the machine interference $M/M/C/K/\infty$ (with balking, reneging and spares)
published in XIX annual conference of the mathematical society, Banaras Hindu University.

2- Response time analysis to the bulk arrival queue system in communication network,
Published in proceeding of Fifth conference of the International academy of Physical Sciences, Bundelkhand University Jhansi.

3- The transient behaviour of the machine interference : M/M/C/K/N (with balking, reneging and spares)

Presented in Ninth annual conference of Gwalior academy of mathematical sciences, Institute of Technology & management sithouli, Gwalior.

4- Transient behaviour of the M/M/C interdependent Queueing model with controllable arrival rates.

Presented in national conference on Information Technology, operations Research and computing, Dau Dayal Institute of vocational education Dr. B.,R. Ambedkar, University, Agra.

5- The steady state analysis of $M/G^x/1$ retrial queue with multiple vacations

Presented in the tenth Annual conference of Gwalior academy of Mathematical sciences, Govt. Kamla Raja Girls post Graduate (Autonomous) college, Gwalior.

This thesis consists of six chapters, each divided into several sections (progressively numbered 1.1, 1.2, ...etc,) The results in the text have been numbered serially, section and chapter wise, e.g. (3.2.1) means first result of second section of chapter third. The references cited have been arranged alphabetically and year wise at the end of thesis wherever more than one paper of the same author occur. The same year, they are distinguished by letters a,b,c.... etc. The references are indicated in the text by giving the year within parenthesis. This work starts with the study of the steady state model of the machine interference with balking, reneging and spares. We analyze two cases of spares first is $c \geq y$ and $c<y$ and also discuss special case.

Chapter I, deals with introduction and brief historical survey of the work, done in the field of Bulk Queueing model, with special references to the work embodied in the thesis.

Chapter II, is devoted to the study of the steady state model of the machine interference, studied by A.I. showky [133]. In this chapter we have analyzed the steady state behaviour of the system and obtained length Queue of each cases.

Chapter III, is devoted to the transient behavior of the machine interference studied by A.I. Shawky [133]. In this chapter we have discuss the transient state behaviour of the system and assume the boundary condition then we have obtained length queue of each cases.

Chapter IV, is devoted to transient behaviour of interdependent queueing model with controllable arrival rates studied by M.I. Aftab Begum and D.Maheswari [107]. In this chapter we have analyzed the transient state behaviour of the system and its laplace transform then we have obtained the Prob. that the system is in state, r and R, empty, conditional Probability state, 1, Mean Queue length and expected waiting and particular cases.

Chapter V, is devoted to the steady state analysis of Bulk Service retrial queue with multiple vacations studied by B. Krishna Kumar and S. Pavai Madneswari [12]. The supplementary variable techniques and boundary condition have been used in this chapter, then we have obtained the probability generating function of the orbit size when server is idle, busy and under repair.

Chapter VI, is to devoted to Response time analysis to the bulk arrival queue system in communication network studied by M.P. Singh V. Kumar and A. Kumar [100]. In this chapter we have analyzed mean waiting time, mean response time and its computation of numerical results.

Ahmad

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Contents

Chapter - 1

INTRODUCTION

Queue is an essential part of life as it has been seen everywhere whenever we face a queue , we have to wait till the clearence of the queue. There are various situations, such as van, post office, Railway reservation counter, Transportation system, Communication system etc. are some of the common queues that we find in our daily routeins. Queueing theory is one of the most important branch of applied probability theory and has vast application in technology and management. As far as the future development of technology and management may be extrapolated from the past and present state of affairs the need for a deeper insight into queueing theory will increase rapidly. It is hardly necessary to point out the actual queueing situations encountered in every day life. Production lines, the theory of scheduling and transportation, the designs of automatic equipment such as telephone and telegraph exchanges and particularly the repidly growing field of information handling and data processing are new fields in which queueing situations are encountered.The beginning of the technological development of automatic telephone exchanges led to infinity number of problems which could be solved in a satisfactory way only by probabilistic method.

It is Erlang (cf. BROCKMEYER, HALSTROM, JENSEN [1948]), A Danish mathematicion, who was the founder of queueing theory. His

studies , in the period of 1909-1920, are now classical work in queueing theory.Until about 1940 the development of queueing theory has mainly been directed by the needs encountered in the design of automatic telephone exchanges. After the second world war, when applications of mathematical models and methods in technology and organizations rose to a level hither to unknown, it was soon recognized that queueing theory has a lot of potential in most applications of wide variety of fields such as, machine maintenance, road traffic, aviation problems, defence operations, inventory management etc. It has been used to study physical phenomena like the semi-conductor noise problem (Bell, 1958), Panico (1969), have given various applications of queueing theory to business, economics and Science.The theory has applications to the following scientific areas, Catalysts in a chemical reaction, filtration process, gas molecules going through a whole, hospitals and the demand for medical Care, nervous reactions, and such psychological stress and strain as neurosis.

In the queueing system the number of units (servers) are available to provide the services for the customers who arrive at their on choice of time. They are served by one or more servers according to a given service discipline, the service times being randomly distributed an that governed by some probabilitstic law and after being served the custmers depart from the system. A queueing system is specified by the following factors:

(1) The Input Process (Arrival Pattern of Coustomers) :- The Input process describes the way in which the customers arrive and join the system. Generally, the customers arrive in a more or less random fashion

which is not worth making the prediction. Thus the arrival pattern can be described in terms of probabilities and consequently the probability destribution for interarrival times (the time between two Successive arrivals) or the distribution of number of customers arriving in unit time must be defined. Another factor of interest concerning the input process is the manner that arrivals come in batches or groups instead of one at a time. In the event where more than one arrival come to the system altogether. The input is said to be bulk or batches type. The situation, where bulk arrival occurs the time between the successive arrivals of the groups also governed by the probabilistic law.

(2) The service Mechanism (service pattern of servers) :- The time interval from the instant of initiation of Service of a customer to the instant when this service ends is called the service time. The service times v_1, v_2----- of the successive units are assumed to be independent of each other and of the Input distribution, and their distribution B (v) is Called the Service time distribution or the holding time distribution.

(If we can write $dA(u)=a(u)\,du$, $dB(v) = b(v)dv$ then a (u) and b (v) are called the input and the service time density functions respectively).

Service may be single or in batch. The service rate may depend on the number of customers waiting for service. A server may work faster if the queue is building up or conversely he may get flustered and become less efficient. The situation in which service depends on the number of customers waiting is referred to as state dependent service.

Even if the service rate is high, in general, customers arrive and depart at irreguler intervals hence the queue length will assume no definite pattern unless arrivals and service are deterministic.

(3) Queue Disicipline :- The queue discipline is the order of manner in which customers in the queue will be served. The most common discipline is called

(i) FCFS- First come first served,

under which customer are served in order of their arrivals.

(II) LCFS - Last come first served

the last arrival in the system is to be provided service first of all.

(III) SIRO - Service in random order

in this queue discipline the arrivals are serviced randomly irrespective of their arrivals in the system.

(iv) PSD - Priority service discipline

under priority discipline, the service is of two types:-

(1) Peeemptive Priority. :- In this case the customer with the highest priority is allowed to enter service immediately after entering into the system even if a customer with lower priority is already in service. That is, lower priority customer's service is interrupted (preempted) to start service for a special customer. This service is resumed again after the higher priority customer is served or according to the preemptive rules.

(2) Non-Preemptive Priority. :- In this case highest priority customer goes ahead of the queue but service is started immediately on completion of the current service.

(4) System capacity. :- In queueing process there is a physical limitation queue of the waiting room, so that when the queue reaches a certain length, no further customer is allowed to enter until space becomes available by a service completion. These are referred to as finite queueing situations,i.e. there is a finite limit to the maximum queue size . A queue with limited waiting room can be viewed as one with forced balking where a customer is forced to balk if he arrives at a time when queue size is at its limit.

(5) The number of channel:- The service facility can have one or more channels queueing processes with single channel is called single channel or single server queueing process, while those with more than one channel is called multichannel or multi server queueing process.

Classification of Queueing Model:- Some of the well known queue models may be classified as ;

(i) (M/M/1): (∞/FC FS):- The queueing model (M/M/1) : (∞ /FCFS) deals with a queueing situation having poisson arrrivals (exponential inter-arrival times) and poisson services (exponential serviee times), single server, infinite capacity of the system and first come first served queue discipline.

(ii) (M/M/1) : (N/FCFS) :- The queueing model (M/M/1):(N/FCFS) is different from the queueing model (M/M/1):(∞ /FCFS) in respect of the Capacity of the system. Here the capacity of the system is limited to N customers in the system while remaiing facts are same.

(iii) (M/M/S) : (∞/FCFS) :- The queueing model (M/M/S) : (∞ /FCFS), as mentioned in this model (M/M/1):(∞/FCFS), while this model instead of a single service channel, this model deals with, multiple servers, in parallel which are to limited S. For this queueing system it is assumed that customers arrive at a mean arrival rate λ and are served according to first come first serve basis at any of these servers and it is also assumed that only one queue is formed and each server has same mean service rate.

(iv) (M/M/S) : (N/FCFS) :- The queueing model(M/M/S) : (N/FCFS) is different from model(M/M/S) : (∞ /FCFS) in respect of the capacity of the system. Here the capacity of the system is limited ot N customers only and first come first served discipline.

(v) (M/M/S) :- (M/GD) Machine servicing Model:- The machine servicing model is same to the model (M/M/S) : (N/FCFS) except in queue discipline, i.e. General Discipline (GD). The system has a finite calling population, of size M (say) and the probability of arrivals depends upon the past behavior.The most common application of the queueing model has been to the machine servicing problem in which say S repairmen (called servers) are assigned the responsibility for maintaining certain group of M machine. In this case, the calling population is the machines and an arrival corresponds to a machine breakdown.

(vi) (M/E$_K$/1):(∞/FCFS):- The queueing model (M/E$_K$/1) : (∞/FCFS) consists of a single service channel in which there are K identical stages (Phases) in series for services i.e. poisson arrivals erlangian service time for K-phases and a single server.

(vii) Power Supply Model :-The power supply model may be used to ensure better utilization of power supply to consumers. Suppose that the demand of power supply from the consumers follow poisson distribution with parmeter and existing supply schedule of power also follows poisson distribuation with parameter μ .

The following terms are used in this thesis :

(i) Busy period distribution :- The busy period is defined as the interval of time from the instant of arrival of a customer at an idle channel to the instant when the channel next becomes idle for the first time .For this type of queue, the initial busy period may start with i>1 customers and hence may be different from other busy periods which starts with the arrival of a single customer. Results for busy periods of single server, single -arrival, poison -input queue can rather easily be generalized to the corresponding bulk arrival queues.For multi- server queues any one of several different definitions of busy period may appropriate.

Any queueing system is described by given first the input process, then the service time distribution and finally number of servers. For example - $M/E_k/1$ stands for a single server queueing system with poisson arrivals and K-Erlang service time distribution and GI / G / S stands for the most general case of general independent input, general service times and S for server.

Prior to 1950, Most of the work had been done under the assumption that the arrival or service time distribution, either exponential or K-Erlang or regular. Very little work had been done for general distribution. Notable

workers before 1950 are Erlang (M/M/S,M/D/S, M/E_k/1), Vaulot (M/M/S), Malina (M/M/S,M/D/S), Palm (M / M/S), crommelin (M/D/S; M/G/S),Khintchine (M/D/1,M/G/1), Pollaczek (M/G/1, E_k/G/1) M/M/S, M/D/s, M/G/S), Borel (M/D/1), and volberg (E_k/G/1) . A review of the work during this period has been given by kendall, Saathy, N.S. Kambo, M.L. Chaudhary and Easton .

(II) SUPPLEMENTARY VARIABLE TECHNIQUES:- This technique was used in 1942 by Kosten this method consists in enhancing the description of the stated of the system by including one or more supplementary variables so as to make the process Markovian. Thus the queueing process M/G/1 is made Markovian by including the variable x, the elapsed serviced time of the unit being serviced ,and define P_n (x,t) dx as the probability that at time t, the queue length is n and the elapsed service time of the unit under service lies between x and x+dx. Similarly the Process GI/M/1 may be dealt with by the inclusion of the elapsed time for the last arrival. Kendall (1953) [97], who called it the augmentation technique. Formulation of the method and mentioned that the most general process GI/G/S can be studied similarly by using s+1 supplementary variables technique denoting respectively the elapsed time since the last arrival and the elapsed service time of each of the s servers.

The name appears due to Cox (1955b) [44] who used a supplementary variable technique to study the queueing system M/G/1, kendall (1953) [97] considered this technique, which he labeled 'augmentation but preferred

the use of an imbedded Markov' chain as leading to simpler calculation. Colony (1958, 1959, 1960b) used this technique in studies of the process of the system GI/M/1 and to study the system $GI/E_k/1$. Jaiswal (1961c, 1962) applied the technique to obtain transient solution of the permeative resume and the head of the line priority queues. The same technique has been used by thiruvengadum (1963, 1964) to study queueing system subject to breakdown by thiruvengadum and Jaiswal (1963, 1964a,1964b) to study Machine Interference problems and by Jaiswal (1965) [89] to study the M/G/1queue with balking and reneging Kashyap (1965a, 1965c, 1967) used the supplementary variable technique to study the transient behaviour of the double ended queue with limited waiting space and in (1966) extending the use of supplementary variable technique has studied head of the line priority queues with bulk arrivals. Kashyap (1972b 1972c) used the supplementary variable technique to study preemptive repeat priority and permeative queueing discipline.

Gaur and Kashyap (1972 h) [69], using the supplementary variable method obtained the joint distribution of the queue later (1972 i) using the same method head of the line priority queueing system with general service has been studied to obtain an expression for the laplace transform of generating function of the joint probability distribution of the number of priority and non- priority units in the queue at time t and the number of units served in time t.

(III) HYPERGEOMETRIC FUNCTIONS :- Hypergeometric function is denoted by ${}_pF_g(\alpha_2,\alpha_2............\alpha_{p,}\ \rho_1\ \rho_2\ ....\rho_g\ ;z)$ or simply pFg when there is no confusion about the arguments of the function:F(a, b ; c ; x) or F [$^{a,\,b}$:c ; x] or ${}_2F_1$ [$^{a,\,b}$:c ; x], Where the suffix 2 in ${}_2F_1$ denotes the number of parameters in the numerator and the right suffix 1 in ${}_2F_1$ denotes the number of parameters in the denominator of the series and the hypergeometric function does not change if the parameters α and β are interchanged keeping γ fixed.

$F(\alpha, \beta ;\gamma ; x) = F(\beta, \alpha ; \gamma ; x)$ and its deduction by definition

$(\alpha)_n = \alpha\,(\alpha+1)\ ....(\alpha + n - 1)$

$(\alpha)_{n+1} = \alpha\,(\alpha+1)_n$

$(\alpha+n) - (\alpha)_n = (\alpha)_{n+1}$

(IV)CONTROLLABLE ARRIVAL RATES :- Generally we consider a single server infinite capacity queuing system with the assumption that the arrival and service processes of the system are correlated and follows a bivariate poisson process. In addition to this interdependence it is also assumed that whenever, the queue size reached a certain prescribed number R, the arrival rate reduces from λ_0 to λ_1 and it continues with that rate as longer as the content in the queue is greater than some other prescribed integer r ($r \geq 0$ and $r \leq R$) when the content reaches R, the arrival rate charges back to λ_0 and the same process is repeated, This type of operating strategy is called controllable arrival rates. In the controllable Arrival rates Lindly (1952) [104] analyzed a single server queueing model with

recurrent input and arbitrary service times with the assumption that the service time of the n^{th} customer and time interval epoch are related by a linear function. Conolly and Hadidi (1969) [38] have studied the single server queueing model with the assumption that the rate S_n/T_n is constant for all n , where S_n is the service time of the n^{th} customer and T_n is the time interval between n^{th} and $(n+1)^{th}$ arrival epoch, Michel and Paulson (1979) [114] have considered an M/M/1 queueing system with an assumption that the inter arrival times separating the arrival from that of his predecessor and the customer service times follows a bivariate exponential distribution. Lengaris (1987) [103] Borst and combe (1992) [25] have studied the busy period analysis of a correlated queue with exponential demand and service.

BULK QUEUE :- The concept of bulk queues was implied in the work of Erlang. The study of queues with bulk service is originated in 1954 by Bailey. During the last four decades, queues with bulk service

system with bulk service and finite capacity. The units arrive at a single service station and have received greater attention from researchers and practical scientists.

A part from having the organized/ systematic mathematical structures, queues with bulk service have been used for modelling various situations like shiploading and unloading at a port dispatching of mail bags at central sorting stations, transhipment of members though elevators and at bus stop etc. In all these models are very interesting to note that the two major constitunt process play a vital role namely.

(i) Arrival Process

(ii) Service Process

QUEUEING SYSTEM WITH BULK SERVICE :- Queues with bulk service are described as customers get service in batches. The size of the batches for service may be fixed or random variables. The service times of different batches are independent and identically distributed random variable. The maximum size of a service group is known as service capacity.

Bailey (1954) [16] studied a queueing process in which customers arrive at random from a single queue in order of arrivals and are served in batches. The size of each batches has a fixed maximum. The Equilibrium distribution of queue length has been studied by an imbedded markov chain technique and its ergodicity was established when average demand is less than to the average supply available. Downton (1955, 1956) developed the waiting time distributions of these models and the limiting behaviour of the queue when the batch size tends to infinity Jaiswal (1960 a, 1960 b, 1961) further studied these queueing models and considered the case when maximum number of units to be taken is not constant but depends on the number of units already present with the service as well as the capacity.

Foster and Nyunt (1961) [58] considered a single server queueing system in which items arrive at the sequence of instants t_1, t_2..........such that inter arrival times are inter independent randon variable with an exponential distribution with consideration that the items are in batches of

exactly k units by a single server with respect to arrival and discuss about the system characteristics of an $E/G^k/1$ queue. Finch (1962) [57] studied the transient behavior of a queueing are served in batches of exactly x units, when x>1. The service times of successive batches are independently and identically distributed random variables with a common distribution.

Foster and perera (1964) [59] discussed about the queue size after a departure and considered the queueing problem earlier discussed by foster & Nyut (1961) [58] under the assumption the units arrive in a poison process with parameter λ and served in batches of fixed size k and the service time distribution being arbitrary. Foster (1964) discussed a general single server queueing system when the units are single arrivals and batch departures as well as batch arrivals and single departures and compared the queue size with various instants that are just before arrivals just after departures and through time.

Bhat (1964) [20] discuss about the bulk service queueing system, the arrivals arriving with a fixed time interval follows a binomials and the service time is a discrete variable assumed to be independently identically distribution. The busy period and the queue lengths are obtained in terms of probability generating function. Aurora (1964) [9] had been studied the bulk service is exponential and obtained time dependent probabilities for the queue length of busy periods for at least one channel being busy and both channels being busy.

Neuts (1965) [119] studied the busy period distribution of the queue in which the customers have been served m at a time. If there are less then m or more present and all at a time if there are less than m present and the units arrival in poisson process.

Kashyap (1966) [94] considered the double ended queueing system for taxis and customer at a taxi stand under the assumption that there is limited waiting space both for customer and taxis and the customers are served in bulk. The arrivals of customers taken as general and taxis as poisson and used the supplementary variable technique in which the probability of the state is empty. The cases of erlangian and poisson arrivals of customer and of single stated taxies are studied as particular cases Roes (1966) studied the batch service Markov chain technique. Goyal (1967) [72] studied the bulk exponential service to obtain the steady state probabilities with generating function.

Ghare (1968) [71] considered poisson arrivals and exponential service under bulk service pattern. This queueing problem is an extension to Aurora (1964) [9] for two channels, by using the generating functions and Laplace transformations and also obtained the transient probabilities and discuss about the characteristics of the system. Love (1970) [106] improved the steady state solution for the queueing system where the arrivals are servied in batches by parallel servers of fixed size. He considered the inter arrival times as erlangian with a certain degree and the service time for all servers are exponentially distributed stewart (1970) [136] considered

M/G/1 batch service queueing system with balking and derived the steady - state probabilities and length of the queue.

Murali (1972a , 1972 b) considered the steady state behaviour of a discrete time, single server queueing problem in which the arrivals at two successive time marks are correlated the service is accomplished in batches and the completion of batches at two successive time marks are correlated through probability generating function. He obtained the number of batches waiting for the service. He also studied the transient behavior of a single server queueing system in which the arrivals follow a poisson distribution of batches of variable size and the output follows a general distribution of batches of variable size. He obtained the waiting line size when the service time distribution is exponential with fixed branches, phase type and exponential. Medhi & Borthokur (1972) [110] analyzed the queueing system under three service mechanism.

(i) The server will start the service only when a minimum number of customers present in the queue.

(ii) If the number of customer in the queue lies between a and b the whole queue is taken into service as a bulk discipline.

(iii) If the number of customers in the queue are more than b, the first b customers will be taken into service.

Deb and serfzo (1973) [47] considered batch size discipline where each batch size and its time of service is controlled. To minimize the expected continuously discounted cost per unit time for service over an infinite time

horizen, they used the markov decision process i.e. dynamic programming approach.

Ernest (1973) [52] considered the maximum and minimum times for service to all Jobs in a batch of random size with unknown number of servers and all the servers have the same service time distribution and also obtained the expected minimum batch service time for exponential, uniform service time distribution and poisson, binomial, Negative Binomial Probability mass functions for batch size.

Villiam Makis (1984) discussed the poisson arrivals and the general service time with a single server in which the capacity is infinite. He derived explicit formula for the long run expected average cost per unit time and properties of the average cost function and also determined the optimal control limit in the general case. Deb (1984) considered the compound poisson arrivals, to control a batch service queueing process B. Powell (1986, 1987) considered a queueing model in which the arrivals and departures are bulk on three strategies. The strategies are cancellation strategy holding strategy and combined vehicle holding and cancellation stretregy. He developed moments of the queue length and derivation of the queue length transform. He also obtained the distribution numerically for waiting times of queues with out control for vehicle cancellation strategy and vehicle holding strategies under the consideration of bulk arrivals and bulk service in the truck load motor carrier industry. The mean and variance of the calculated distribution is strongly related to analytical expressions.

Inter dependent Queueing Models : - Bhat (1969) [21] Classified the dependent queues into five categories. They are :

(i) Systems with correlated arrivals

(ii) Systems with correlated services

(iii) Systems with state dependent arrivals.

(iv) Systems with state dependent service

(v) Systems with interdependent arrival and service process.

Jacob and Patricia (1980) [81] considered the queueing models for a sequence of correlated exponential inter arrival and service times for a single server queue shun Cheii Niu (1981) analyzed a M/G/1 queueing system in which the service time required by a customer is dependent on the inter arrival time between his arrival and that of his predecessor using the concept of random variables developed by Esary. Prochan and wolkup (1967), Garg and khanna (1983) considered the steady state behaviour of a queueing system with queue length depending on additional server facility.

Rao (1986) [126] analyzed a M / M /1 queueing system based on bivariate poisson distribution with interdependent arrivals and services & the system behavior through dependence parameter and obtained explicit expressions. Rao (1989) [125] extended these models to inite source queueingmodel and queues with impatience. Gupta (1990) study a first come first served semi markov queue in which both the arrival and service mechanism are semi markov processes. The inter arrival and service times may depend on one another and the marginal service times is assumed to

be phase type the waiting time is spent in the system and virtual waiting time are obtained and analyzed.

Rao (1991) considered the interdependent queueing models for analyzing the situations arising at communication networks. Rao (1983) developed interdependent queueing models for studying the performance measures of the resource sharing systems in computer and also obtained the optimal operating policies for these interdependent. queueing models having Erlangian service time using the dependence structure given by Rao (1986). Borst et al (1993) [26] had been studied the M/G/1 queue in which the service times of arriving customers depend on the length of the interval between their arrival and previous arrival.

$M/M^x/1$ Interdependent queueing model with Varying Batch Size :-

Murty (1993) extended the single server interdependent queueing model to bulk service model. He consider a different problem which is a generalization of the earlier queueing models. Two variations are considered as follows :

(i) The customer are served x at a time, except when less then x are in the system and ready for service at which all units are served and

(ii) The batch size must be exactly x and if it is not, then the server remains idle until the queue size reached x, A typical situation of this model is transportation of men and material are the customer. For this short of situation in order to have optimal transhipment, it is appropriate to approximate the situations with the model having the interdependent arrival and service

process. For developing these interdependent models with bulk service rule we make use of the dependence structure given by Rao (1986) [126].

Let $P_n(t)$ be the probability that there are n customers in the system at time t. The difference differential equations of the model are -

$$P_n^{I}(t) = -(\lambda + \mu - 2e)\, P_n(t) + (\lambda - e)\, P_{n-1}(t) + (\mu - e)\, P_{n-k}(t) \quad n > 1 \quad (1.1)$$

$$P_0^{I}(t) = -(\lambda - e)\, P_0(t) + (\mu - e) \sum_{i=1}^{k} P_i(t) \qquad (1.2)$$

Where λ and μ are marginal arrival rate, Service rate and mean dependence rate respectively.

Consider that the system reached the steady state, the transition equation of the model are:-

$$-(\lambda + \mu - 2e)\, P_n + (\lambda - e)\, P_{n-1} + (\mu - e)\, P_{n-k} = 0$$

$$-(\lambda - e)\, P_0 + (\mu - e) \sum_{i=1}^{k} P_i = 0$$

Using the heuristic arguments of Gross and Harris (1974). They obtained the solution of these equations as

$$P_n = cr^n \quad n > o, \; o < r < 1 \qquad (B)$$

where r is the root which lie in (0,1) of the characteristic equations

$$[(\mu - e)\, D^{k+1} - (\lambda + \mu - 2e)\, D + (\lambda - e)]\, P_n = 0$$

Here D is the differential operator.

The probability that the system is empty is

$$P_0 = (1 - r) \qquad (A)$$

Where r is as given in equations (A)

M/M^x/1 Interdependent Queueing Model With Fixed Batch Size :-

Murty (1993) [117] Considered a slight variation of the model. Along with the other assumption, hence he assumed that the batch size must be exactly x, and if not, then the server waits until such time to start. in another case to reduce the idle time , it may be feasible to use interdependence relation for such type of models He assumed that the number of arrivals and the service of any batch are correlated and follows a bivariate poisson process with the dependence structure the difference differential equations of the model are :-

$$P^{I}_{n}(t) = (\mu - e)\, P^{I}_{n+k}(t) - (\lambda - \mu + 2e)\, P^{I}_{n}(t) + (\lambda - e)\ P^{I}_{n-1}(t), n \geq k \quad (1.3)$$

$$P^{I}_{n}(t) = (\mu - e)\, P^{I}_{n+k}(t) - (\lambda - e)\, P^{I}_{n}(t) + (\lambda - e)\ P^{I}_{n-1}(t), 1 \leq n < k \quad (1.4)$$

$$P^{I}_{0}(t) = = -(\lambda - e)\, P^{I}_{0}(t) + (\mu - e)\, P^{I}_{n}(t), n = 0 \quad (1.5)$$

$$V = \frac{k^2-1}{12} + \frac{r}{(1-r)^2}$$

Where r is as given in equation(B)

The cofficient of variation of the model is

$$C.V. = \frac{(1-r)\,[\,(k^2 - 1)\,(1 - r) + 12r]}{3\,[\,(k - 1)\ \ (1 - r) + 2r]} \quad (1.6)$$

Where r is as given in equation

M/M^x/1 Interdependent Queueing model with variable service capacity :- In the bulk service queuing models Bailey (1954) [16] and Jaiswal (1960) [83] Assumed the model in which the units arrive at random from a single queue in order of arrival and are served in batches. The size

of each batch being either a fixed number of customer or the whole queue length. Which ever is smaller. Jaiswal (1961) [85] extended this model to this case, where at a service epoch m, ($o \leq m \leq s$) if m persons are already present with the server then (s-m) persons or whole queue length. Which is smaller will be taken into service. The service rule is termed as Bailey bulk service rule . However in these models the arrivals and service processes are poissonian and Erlangian respectively with interdependent arrival and service process.

In both the models the system behaviour is analyzed by obtaining the difference differential equations of the models and solving them through generating function techniques. The system characteristics like mean queue length. Variability of the system size, coefficient of variations are derived and analyzed in the light of the dependence parameter, these models are included in earlier models as particular cases for specific values of the parameters.

Murty (1993) [117] considered the single server system with interdependence arrival and service process having the Bailey's is bulk service with variable capacity Under the present study it is assumed that the server servers only at instants t_1, t_2.........t_n.......... if m, ($0 \leq m \leq S$) person are present in the waiting line at time t_n then the server takes a batch of (s - m) persons or whole queue length which ever is smaller, where s is the service capacity.

Two server interdependent queue model with Bulk of Bailey's service :- Murty (1993) [117] considered two server independent bulk queueing model with Bailey's service rule for analysis. The model is an extension of the models considered earlier or two server case. it is also considered that the bulk service process having two service facilities with capacity b_1 and b_2 . A batch of b_1 units or the whole queue length whichever is smaller is taken from the head of the queue for service in the first channel when ever it is free similarly the second channel on becoming free takes $b_2 \leq b_1$ or the whole queue length which ever is less.

If both the servers are idle and there is no queue the next unit to arrive always goes to the first service facility. Further we assumed that the two service facilities are independent of each other and the arrival and service process are independent.

This sort of situation are more common in Marshalling yard's with two engines. Elevator process with two lifts etc. This model is also extended to multiple server independent queueing models with Bailey's Bulk service .In both the models he first develop the difference differential equation and solved them through generating function techniques.

Queueing system with Bulk Arrival :- The study of bulk have begun with Erlang solution of the $M/E_k/1$ Since it gives implicit solution of $M^K/M/1$. Explicit consideration of bulk arrival queues seems to have begun after several years of the work of Bailey (1954) [16] on bulk service In "ordinary" queueing problems it is assumed that customers arrive singly

at a service facility. However this assumption is violated in many real - world queueing situation. Letters arriving at a post office, ships arriving at a port in convey, people going to a theatre, restaurant, and so on are some of the examples of queueing situations where customers do not arrive one by one in bulk or groups, where the size of an arriving groups may be a fixed number i.e. Bulk or batch operations are common on the service side of some system. ,

A set of chapmen-kolmogorov (differential -difference) equations is

$$\frac{d\,P_n(t)}{dt} = -(\lambda+\mu)\,P_n(t) + \mu P_{n+1}(t) + \lambda \sum_{K=1}^{n} P_{n-K}(t)\,C_K \qquad n \geq 1 \qquad (1.7)$$

$$\frac{d\,P_n(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) \qquad (1.8)$$

Chapter - 2

The Steady state Model of the Machine Interference $M/M/C/K/\infty$ (with balking, reneging and spares)

INTRODUCTION :

This chapter consists the queuing model $M/M/C/K/\infty$ with balking, reneging and spares. C servers are available, customers arriving rate is λ. We take Y spares so that when a machine fails, a spare is immediately substituted for it, if it happens that all spares are used and a breakdown occurs then the systems becomes short. When a machine is repaired it then becomes a spares (unless the system is short in which case the repaired machine goes immediately into service). We analyze the two cases of spares, first of them is $C \leq Y$ and the other $C > Y$.

Kleinrock [98] had worked out on the truncated poisson distribution for solving the problem M/M/C/K/N for machine interference but without assumptions of balking, reneging or spares. Gross and Harris [75] discussed the system : M/M/C/m/m with spares only and Medhi [112] treated the system M/M/C/m/m without assumption of balking, reneging and spares. Showky [133] considered M/M/C/K/N machine interference model with balking, reneging & spares with the development of technology of manufacturing, such as those found in computer and communication system, the efficiency of machine system is critical to overall competitiveness. Many researchers have been worked on machine repair problem.

Gupta and Rao [76] derived a recursive method to compute the steady state probabilities of M/G/1 machine interference model Jain *et. al.* extended the work of Shawky [132] by including additional repairman in case of long queue of failed units.

The main assumption of present chapter is an infinite source o infinite customers. Each with an arriving rate λ, c servers are available, that the service times are identically exponential random variables with rate μ also the system has finite storage room and also assume that we have Y spares on hand. So that when a machine fails, a spares is immediately substituted for it if it happens that all spares are used and a breakdown occurs, then the system becomes short. When a machine is repaired, it then becomes a spares (unless the system is short in which case the repaired machine goes immediately into service). Here we analyze the two cases of spares first of them is $C \geq Y$ and $C < Y$ and also discuss special case.

2.1 Case (I)

In this case we study the maching interference system : M/M/C/K/ ∞ with balking, reneging and spares $Y \geq C$. Then the set of birth death coefficients are as follows :

$$\lambda_n = \begin{cases} \lambda & 0 \leq n < c \\ \beta\lambda & c \leq n < Y \\ (n-Y+1)\beta\lambda & Y \leq n < Y+k \\ 0 & n \leq 0 \end{cases}$$

and

$$\mu_n = \begin{cases} n\mu & 0 \leq n \leq c \\ c\mu+(n-c)\alpha & c+1 \leq n < Y+k \end{cases}$$

In the usual arguments of the δ- technique the probabiliy differential difference equations are :

$$P'_n(t) = -\lambda P_0(t) + \mu P_1(t) \quad n=0 \tag{2.1.1}$$

$$P'_n(t) = -(\lambda+n\mu)P_n(t)+\lambda P_{n-1}(t)+(n+1)\mu P_{n+1}(t) \quad 1 \leq n < c \tag{2.1.2}$$

$$P'_c(t) = -(\beta\lambda+c\mu)P_c(t)+\lambda P_{c-1}(t)+(c\mu+\alpha)P_{c+1}(t) \quad n=c \tag{2.1.3}$$

$$P'_n(t) = -[\beta\lambda+c\mu+(n-c)\alpha]P_n(t)+\beta\lambda P_{n+1}(t)+[c\mu+(n+1-c)\alpha]P_{n+1}(t) \quad c+1 \leq n < Y \tag{2.1.4}$$

$$P'_n(t) = -[(n-Y+1)\beta\lambda+c\mu+(n-c)\alpha]P_n(t)+(n-Y)\beta\lambda P_{n-1}(t)+[c\mu+(n+1-c)\alpha]P_{n+1}(t) \quad Y+1 \leq n < Y+K \tag{2.1.5}$$

As $t \to \infty$, the steady - state probability - difference equations are. :-

$$-\lambda P_0 + \mu P_1 = 0 \qquad n=0 \qquad (2.1.6)$$

$$-(\lambda + n\mu) P_n + \lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \qquad 1 \le n < c \qquad (2.1.7)$$

$$-(\beta\lambda + c\mu) P_c + \lambda P_{c-1} + (c\mu + \alpha) P_{c+1} = 0 \qquad n = c \qquad (2.1.8)$$

$$-[(\beta\lambda + c\mu + (n-c)\alpha] P_n + \beta\lambda P_{n-1} + [c\mu + (n+1-c)\alpha] P_{n+1} = 0$$

$$c+1 \le n < Y \qquad (2.1.9)$$

$$-[(n-Y+1)\beta\lambda + c\mu + (n-c)\alpha] P_n + (n-Y)\beta\lambda P_{n-1} + [c\mu + (n+1-c)\alpha] P_{n+1} = 0$$

$$Y+1 \le n < Y+K \qquad (2.1.10)$$

From equation. (2.2..6)

$$-\lambda P_0 + \mu P_1 = 0 \qquad n=0$$

$$\mu P_1 = \lambda P_0$$

$$P_1 = \frac{\lambda}{\mu} P_0$$

$P_1 = \rho P_0$ Where $\rho = \lambda / \mu$

From equation. (2.1.7)

$$-(\lambda + n\mu) P_n + \lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \qquad 1 \le n < c$$

$$(n+1)\mu P_{n+1} = (\lambda + n\mu) P_n - \lambda P_{n-1}$$

Put n = 1

$$2\mu P_2 = (\lambda + \mu) P_1 - \lambda P_0$$

$2 P_2 = (\rho + 1) P_1 - \rho P_0$ Where $\rho = \lambda / \mu$

$$2 P_2 = (\rho + 1)\rho P_0 - \rho P_0$$

$$2 P_2 = \rho P_0 (\rho + 1 - 1)$$

$$2 P_2 = \rho^2 P_0$$

$$P_2 = \frac{\rho^2}{2!} P_0 \qquad \text{i.e} \quad P_2 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 P_0$$

Put n = 2

$$3\mu P_3 = (\lambda + 2\mu) P_2 - \lambda P_1$$

$$3 P_3 = (\rho + 2) P_2 - \rho P_1$$

$$= (\rho + 2) \frac{1}{2!}\rho^2 P_0 - \rho.\rho P_0$$

$$= (\rho + 2) \frac{\rho^2}{2!} P_0 - \frac{\rho^2}{2!} 2 P_0$$

$$= \frac{1}{2!}\rho^2 P_0 [\rho + 2 - 2]$$

$$3 P_3 = \frac{1}{2!} \rho^3 P_0$$

$$P_3 = \frac{1}{3!}\rho^3 P_0$$

$$P_3 = \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 P_0$$

Put n = 3

$$4\mu P_4 = (\lambda + 3\mu) P_3 - \lambda P_2$$

$$4 P_4 = (\rho + 3) P_3 - \rho P_2$$

$$= (\rho+3) \frac{1}{3!} \rho^3 P_0 - \rho \frac{1}{2!} \rho^2 P_0$$

$$= (\rho + 3) \frac{1}{3!} \rho^3 P_0 - \frac{1}{2!} \rho^3 P_0$$

$$= (\rho+3) \frac{1}{3!} \rho^3 P_0 - \frac{1}{3!} 3 \rho^3 P_0$$

$$= \frac{1}{3!} \rho^3 P_0 [\rho + 3 - 3]$$

$$4 P_4 = \frac{1}{3!} \rho^4 P_0$$

$$P_4 = \frac{1}{4!} \rho^4 P_0$$

Put n = 4

$$5\mu\, P_5 = (\lambda + 4\mu)\, P_4 - \lambda\, P_3$$

$$5\, P_5 = (\rho + 4)\, P_4 - \rho\, P_3$$

$$= (\rho + 4)\, \frac{1}{4!}\rho^4\, P_0 - \rho\, \frac{1}{3!}\, \rho^3\, P_0$$

$$= (\rho + 4)\, \frac{1}{4!}\, \rho^4 P_0 - \frac{1}{4!}\, \rho^4\, P_0\, (4)$$

$$= \frac{1}{4!}\, \rho^4\, P_0\, [\rho + 4 - 4]$$

$$P_5 = \frac{1}{5!}\, \rho^5\, P_0$$

Similarly

$$P_n = \frac{1}{n!}\, \rho^n P_0 \text{ Where } \rho = \frac{\lambda}{\mu}$$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \quad 0 \le n < c \qquad (2.1.11)$$

Let $\gamma = \lambda/\alpha$ and $\rho = c\ \mu/\alpha$

From equation. (2.1.8)

$$-(\beta\lambda + c\mu)\,P_c + \lambda\,P_{c-1} + (c\mu + \alpha)\,P_{c+1} = 0 \quad n = c$$

$$-(\beta\gamma + \delta)\,P_c + \gamma\,P_{c-1} + (\delta + 1)\,P_{c+1} = 0$$

$$(\delta + 1)\,P_{c+1} = (\beta\gamma + \delta)\,P_c - \gamma\,P_{c-1}$$

$$P_{c+1} = \left(\frac{\beta\gamma + \delta}{\delta + 1}\right) P_c - \frac{\gamma}{\delta + 1}\,P_{c-1}$$

$$P_{c+1} = \left(\frac{\beta\gamma}{\delta + 1}\right) P_c + \left(\frac{\delta}{\delta + 1}\right) P_c - \frac{\gamma}{\delta + 1}\,P_{c-1}$$

$$P_{c+1} = \left(\frac{\beta\gamma}{\delta + 1}\right)\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0 + \left(\frac{\delta}{\delta + 1}\right)\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0 - \left(\frac{\gamma}{\delta + 1}\right)\frac{1}{(c-1)!}\left(\frac{\lambda}{\mu}\right)^{c-1} P_0$$

$$= \left(\frac{\beta\gamma}{\delta + 1}\right)\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0 + \left(\frac{\delta}{\delta + 1}\right)\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0 - \left(\frac{\lambda}{\alpha(\delta + 1)}\ \frac{1}{(c-1)!}\left(\frac{\lambda}{\mu}\right)^{c-1} P_0\right)$$

$$= \left(\frac{\beta\gamma}{\delta + 1}\right)\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0 + \left(\frac{\delta}{\delta + 1}\right)\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0 - \frac{\lambda}{\alpha(\delta + 1)}\ \frac{1}{(c-1)!c}\ \frac{\left(\frac{\lambda}{\mu}\right)^{c-1}\left(\frac{\lambda}{\mu}\right)^c}{\left(\frac{\lambda}{\mu}\right)} P_0$$

$$= \left(\frac{\beta\gamma}{\delta + 1}\right)\frac{1}{c!}\rho^c P_0 + \left(\frac{\delta}{\delta + 1}\right)\frac{1}{c!}\rho^c P_0 - \frac{\rho^c}{c!}\ \frac{\lambda c \mu}{\alpha(\delta + 1)\lambda} P_0$$

$$= \left(\frac{\beta\gamma}{\delta + 1}\right)\frac{1}{c!}\rho^c P_0 + \left(\frac{\delta}{\delta + 1}\right)\frac{1}{c!}\rho^c P_o - \left(\frac{\delta}{\delta + 1}\right)\frac{\rho^c}{c!} P_0$$

$$P_{c+1} = \left(\frac{\beta\gamma}{\delta + 1}\right)\frac{1}{c!}\rho^c P_0$$

$$P_{c+1} = \left(\frac{\beta\gamma}{\delta + 1}\right)\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0$$

From equation. (2.1..9)

$$-(\beta\lambda + c\,\mu + (n-c)\alpha)\,P_n + \beta\lambda P_{n-1} + [c\,\mu + (n+1-c)\alpha]\,P_{n+1} = 0$$

$$c+1 \le n < Y$$

$$-(\beta\gamma + \delta + (n-c))P_n + \beta\gamma\,P_{n-1} + [\delta + n + 1 - c]\,P_{n+1} = 0$$

$$[\delta + n + 1 - c]\,P_{n+1} = [\beta\gamma + \delta + (n-c)]\,P_n - \beta\gamma\,P_{n-1}$$

Put n = c+1

$$[\delta + c + 1 + 1 - c]\,P_{c+2} = [\beta\gamma + \delta + c + 1 - c]\,P_{c+1} - \beta\gamma\,P_c$$

$$(\delta+2)\,P_{c+2} = (\beta\gamma + \delta + 1)\,P_{c+1} - \beta\gamma\,P_c$$

$$(\delta+2)\,P_{c+2} = (\beta\gamma + \delta + 1)\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left(\frac{\beta\gamma}{\delta+1}\right) P_0 - \beta\gamma\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c P_0$$

$$(\delta+2)\,P_{c+2} = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \beta\gamma\,P_0 \left[\frac{\beta\gamma + \delta + 1}{\delta+1} - 1\right]$$

$$= \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \beta\gamma\,P_0 \left(\frac{\beta\,\gamma}{\delta+1}\right)$$

$$P_{c+2} = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\,\gamma)^2}{(\delta+1)\,(\delta+2)}\,P_0$$

Put n = c+2

$$[\delta + c + 2 + 1 - c]\,P_{c+3} = [\beta\gamma + \delta + c + 2 - c]\,P_{c+2} - \beta\gamma\,P_{c+1}$$

$$(\delta+3)\,P_{c+3} = (\beta\,\gamma+\delta+2)\,P_{c+2}-\beta\,\gamma\,P_{c+1}$$

$$= (\beta\gamma+\delta+2)\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\gamma)^2}{(\delta+1)\,(\delta+2)}\,P_0-\beta\gamma\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left(\frac{\beta\gamma}{\delta+1}\right)P_0$$

$$= \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\gamma)^2}{\delta+1}\,P_0\left[\frac{\beta\gamma+\delta+2}{\delta+2}\ -\ 1\right]$$

$$P_{c+3} = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\gamma)^3}{(\delta+1)\,(\delta+2)\,(\delta+3)}\,P_0$$

Put n = c + 3

$$\left(\delta+c+3+1-c\right]P_{c+4} = \left[\beta\gamma+\delta+c+3-c\right]P_{c+3}-\beta\gamma\,P_{c+2}$$

$$(\delta+4)\,P_{c+4} = (\beta\gamma+\delta+3)\,P_{c+3}-\beta\gamma\,P_{c+2}$$

$$= (\beta\gamma+\delta+3)\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\gamma)^3\,P_0}{(\delta+1)\,(\delta+2)\,(\delta+3)}\,-$$

$$\beta\gamma\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\gamma)^2}{(\delta+1)\,(\delta+2)}\,P_0$$

$$= \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\gamma)^3}{(\delta+1)\,(\delta+2)}\,P_0\,\frac{[\beta\gamma+\delta+3}{\delta+3}\ -1]$$

$$P_{c+4} = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\gamma)^4}{(\delta+1)\,(\delta+2)\,(\delta+3)\,(\delta+4)}\,P_0$$

Put n = c + 4

$$(\delta+c+4+1-c)\,P_{c+5} = (\beta\gamma+\delta+c+4-c]\,P_{c+4}-P_{c+3}\,\beta\gamma$$

$$(\delta+5)\,P_{c+5} = (\beta\gamma+\delta+4)\,P_{c+4}-\beta\gamma P_{c+3}$$

$$= (\beta\gamma+\delta+4)\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\gamma)^4}{(\delta+1)\,(\delta+2)\,(\delta+3)\,(\delta+4)}\,P_0\,-$$

$$\beta\gamma\,\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{(\beta\gamma)^3}{(\delta+1)\,(\delta+2)\,(\delta+3)}\,P_0$$

$$= \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^{c} \frac{(\beta\gamma)^{4}}{(\delta+1)(\delta+2)\,(\delta+3)}\, P_0 \left[\frac{\beta\gamma+\delta+4}{\delta+4} \;-1\right]$$

$$P_{c+5} = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^{c} \frac{(\beta\gamma)^{5}}{(\delta+1)\,(\delta+2)(\delta+3)(\delta+4)\,(\delta+5)}\, P_0$$

Similarly

$$P_n = P_{c+(n-c)} = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^{c} \frac{(\beta\gamma)^{n-c}}{(\delta+1)_{n-c}}\, P_0 \qquad c+1 \le n < Y \quad (2.1.12)$$

From equation. (2.1.10)

$$-\left[(n-Y+1)\,\beta\lambda + c\mu + (n-c)\alpha\right] P_n + (n-Y)\,\beta\lambda P_{n-1} +$$

$$\left[c\mu + (n+1-c)\alpha\right] P_{n+1} = 0 \quad Y+1 \le n < Y+k$$

$$(\delta+n+1-c)\, P_{n+1} = \left[(n-Y+1)\,\beta\gamma + \delta + n - c\right] P_n -$$

$$(n-Y)\,\beta\gamma P_{n-1}$$

Put n $= Y+1$

$$(\delta+Y+1+1-c)\, P_{Y+2} = \left[(Y+1-Y+1)\,\beta\gamma + \delta + Y + 1 - c\right] P_{Y+1}$$

$$-\ (Y+1-Y)\,\beta\gamma P_Y$$

$$(\delta+Y+2-c)\, P_{Y+2} = \left[2\,\beta\gamma + \delta + Y + 1 - c\right] P_{Y+1} - \beta\gamma P_Y$$

$$= (2\,\beta\gamma + \delta + Y + 1 - c)\, \frac{1}{c!}\,\rho^{c}\, \frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\, P_0 \;-$$

$$\beta\gamma\, \frac{\rho^{c}}{c!}\, \frac{(\beta\gamma)^{Y-c}}{(\delta+1)_{Y-c}}\, P_0$$

$$= \frac{1}{c!}\rho^{c}\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c}}\,P_0\left[\frac{2\,\beta\gamma+\delta+Y+1-c}{\delta+Y+1-c}\; -1\right]$$

$$= \frac{\rho^{c}}{c!}\,\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c}}\,P_0\,\frac{2\,\beta\gamma}{(\delta+Y+1-c)}$$

$$P_{Y+2} = \frac{\rho^{c}}{c!}\,\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c}}\,\frac{2\,\beta\gamma}{(\delta+Y+1-c)\,(\delta+Y+2-c)}\,P_0$$

$$P_{Y+2} = \frac{\rho^{c}}{c!}\,\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\,\frac{2\,\beta\gamma}{(\delta+Y+2-c)}\,P_0$$

$$P_{Y+2} = \frac{\rho^{c}}{c!}\,\frac{(\beta\gamma)^{Y-c}}{(\delta+1)_{Y-c}}\,\frac{2(\,\beta\gamma)^{2}}{(\delta+1+Y-c)\,(\delta+Y+2-c)}\,P_0$$

Put n = Y +2

$$(\delta+Y+2+1-c)\,P_{Y+3} = \left[(Y+2-Y+1)\,\beta\gamma+\delta+Y+2-c\right]P_{Y+2} - (Y+2-Y)\,\beta\gamma\;P_{Y+1}$$

$$(\delta+Y+3-c)\,P_{Y+3} = \left[3\,\beta\gamma+\delta+Y+2-c\right]P_{Y+2} - 2\,\beta\gamma\;P_{Y+1}$$

$$= (3\,\beta\gamma+\delta+Y+2-c)\,\frac{\rho^{c}}{c!}\,\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\,\frac{2\,\beta\gamma}{(\delta+Y+2-c)}\,P_0$$

$$-2\,\beta\gamma\;\frac{\rho^{c}}{c!}\,\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\,\frac{2\,\beta\gamma}{(\delta+Y+2-c)}\,P_0$$

$$= \frac{\rho^{c}}{c!}\,\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\,2\,\beta\gamma\,P_0\left[\frac{3\,\beta\gamma+\delta+Y+2-c}{\delta+Y+2-c}\; -1\right]$$

$$= \frac{\rho^{c}}{c!}\,\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\,2\,\beta\gamma\,P_0\left(\frac{3\,\beta\gamma}{\delta+Y+2-c}\right)$$

$$= \frac{\rho^{c}}{c!}\,\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\,\frac{2.3\,(\beta\gamma)^{2}}{(\delta+Y+2-c)}$$

$$P_{Y+3} = \frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c+1}\ 2.3.\ (\beta\gamma)^2}{(\delta+1)_{Y-c+1}\ (\delta+Y+2-c)\ (\delta+Y+3-c)} P_0$$

Put n = $Y+3$

$$(\delta+Y+4+1-c)\ P_{Y+4} = \left[(Y+3-Y+1)\ \beta\gamma+\delta+Y+3-c\right] P_{Y+3}$$

$$-\ (Y+3-Y)\ \beta\gamma\ P_{Y+2}$$

$$= \left[4\ \beta\gamma+\delta+Y+3-c\right] \frac{\rho^c}{c!\ (\delta+1)_{Y-c+1}} \frac{(\beta\gamma)^{Y-c+1}\ 2.3.\ (\beta\gamma)^2\ P_0}{(\delta+Y+2-c)\ (\delta+Y+3-c)}$$

$$-\ 3\ \beta\gamma \frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}} \frac{2\ \beta\gamma}{(\delta+Y+2-c)} P_o$$

$$= \frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c+1}\ 2.3.(\beta\gamma)^2}{(\delta+1)_{Y-c+1}\ (\delta+Y+2-c)} P_0 \left[\frac{4\ \beta\gamma+\delta+Y+3-c}{\delta+Y+3-c}\ -1\right]$$

$$= \frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c+1}\ 2.3.(\beta\gamma)^2}{(\delta+1)_{Y-c+1}\ (\delta+Y+2-c)} P_0 \frac{4\ \beta\gamma}{\delta+Y+3-c}$$

$$P_{Y+4} = \frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c+1}\ 2.3.4.(\beta\gamma)^3}{(\delta+1)_{Y-c+1}\ (\delta+Y+2-c)\ (\delta+Y+3-c)} P_0$$

$$P_{Y+4} = \frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c}\ 2.3.4.(\beta\gamma)^4}{(\delta+1)_{Y-c}\ (\delta+Y+1-c)\ (\delta+Y+2-c)\ (\delta+Y+3-c)\ (\delta+Y+4-c)}$$

Similarly,

$$P_n = P_{Y+(n-Y)} = \frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c}\ (n-Y)!\ (\beta\gamma)^{n-Y}}{(\delta+1)_{Y-c}\ (\delta+Y+1-c)_{n-Y}} P_0 \quad (2.1.13)$$

Thus P_n can be written as follows.

$$
P_n = \begin{cases}
\dfrac{1}{n!}\rho^n \, P_0 & 0 \le n < c \quad \text{Where } \rho = \lambda/\mu \\[2ex]
\dfrac{1}{c!}\left(\dfrac{\lambda}{\mu}\right)^c \dfrac{(\beta\gamma)^{n-c}}{(\delta+1)_{n-c}} \, P_0 & c \le n < Y \\[2ex]
\dfrac{\rho^c}{c!} \dfrac{(\beta\gamma)^{Y-c}}{(\delta+1)_{Y-c}} \dfrac{(n-Y)!\,(\beta\gamma)^{n-Y}}{(\delta+Y+1-c)_{n-Y}} \, P_0 & Y+1 \le n < Y+k
\end{cases} \qquad (2.1.14)
$$

To find P_0, the boundary condition, consider $\sum_{n=0}^{\infty} P_n = 1$

$$
P_0\left[\sum_{n=0}^{c-1} \frac{1}{n!}\rho^n + \frac{1}{c!}\rho^c \sum_{n=c}^{Y} \frac{(\beta\gamma)^{n-c}}{(\delta+1)_{n-c}} + \right.
$$

$$
\left. \frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c}}{(\delta+1)_{Y-c}} \sum_{n=Y+1}^{\infty} \frac{(n-Y)!\,(\beta\gamma)^{n-Y}}{(\delta+Y+1-c)_{n-Y}} \right] = 1 \qquad \text{(A)} \qquad (2.1.15)
$$

We now consider

$$
(I) \sum_{n=c}^{Y} \frac{(\beta\gamma)^{n-c}}{(\delta+1)_{n-c}} \qquad n \to n+c
$$

$$
\sum_{n=0}^{Y-c} \frac{(\beta\gamma)^n \; n!}{(\delta+1)_n \; n!}
$$

$$
= \sum_{n=0}^{Y-c} \frac{(\beta\gamma)^n \; (1)_n}{(\delta+1)_n \; n!}
$$

$$
= \; {}_1F_1(1;\ \delta+1;\ \beta\gamma)
$$

$$
(II) \; \frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c}}{(\delta+1)_{Y-c}} \sum_{n=Y+1}^{\infty} \frac{(n-Y)!\,(\beta\gamma)^{n-Y}}{(\delta+Y+1-c)_{n-Y}}
$$

$$
\frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c}}{(\delta+1)_{Y-c}} \sum_{n=0}^{\infty} \frac{(n+1)!\,(\beta\gamma)^{n+1}}{(\delta+Y+1-c)_{n+1}} \qquad n \to n+Y+1
$$

$$
\frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c}}{(\delta+1)_{Y-c}} \sum_{n=0}^{\infty} \frac{(n+1)n!\;\beta\gamma\;(\beta\gamma)^n}{(\delta+Y+1-c)(\delta+Y+2-c)_n}
$$

$$
\frac{\rho^c}{c!} \frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}} \sum_{n=0}^{\infty} \frac{(n+Y)n!\;\;(\beta\gamma)^n}{(\delta+Y+2-c)_n}
$$

$$\frac{\rho^c}{c!}\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\sum_{n=0}^{\infty}\frac{(n+1)!\quad(\beta\gamma)^n}{(\delta+Y+2-c)_n}$$

$$\frac{\rho^c}{c!}\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\sum_{n=0}^{\infty}\frac{(2)_n\quad(\beta\gamma)^n\ n!}{(\delta+Y+2-c)_n\ n!}$$

$$\frac{\rho^c}{c!}\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\sum_{n=0}^{\infty}\frac{(2)_n\quad(\beta\gamma)^n\ (1)n}{(\delta+Y+2-c)_n\ n!}$$

$$\frac{\rho^c}{c!}\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\ {}_2F_1\,(1,2;\delta+Y+2-c;\beta\gamma)$$

Put in (A) we get.

$$P_0^{-1}=\sum_{n=0}^{c-1}\frac{\rho^n}{n!}+\frac{\rho^c}{c!}\ {}_1F_1\,(1;\delta+1;\beta\gamma)+$$

$$\frac{\rho^c}{c!}\frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\ {}_2F_1\,(1,2;\delta+Y+2-c;\beta\gamma)\qquad(2.1.16)$$

Where $(1)_n = 0$ where $n > Y - c$ in the first hypergemetric function.

$(n+1)! = (2)_n$

To calculated the expected number of units in the system, a result due to Abou - E1 - Ata is used, which states that for a simple birth death process.

$$L=\lambda P_0\frac{\partial}{\partial\lambda}P_0^{-1}$$

$$=\lambda P_0\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{c-1}\frac{\rho^n}{n!}+\frac{\rho^c}{c!}{}_1F_1\,(1;\delta+1;\beta\gamma)+\right.$$

$$\left.\frac{\rho^c(\beta\gamma)^{Y-c+1}}{c!\,(\delta+1)_{Y-c+1}}\ {}_2F_1\,(1,2;\delta+Y+2-c;\beta\gamma)\right]$$

$$\text{(I) }\lambda P_0\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{c-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]$$

$$\lambda P_0\left[\sum_{n=0}^{c-1}\frac{1}{n!}\ \frac{n\lambda^{n-1}}{\mu^n}\right]$$

$$P_0\left[\sum_{n=0}^{c-1}\frac{1}{(n-1)!}\ \frac{\lambda^n}{\mu^n}\right]$$

$$P_0 \sum_{n=0}^{c-1} \frac{1}{(n-1)!} \frac{\lambda^n}{\mu^n}$$

$$P_0 \sum_{n=0}^{c-1} \frac{1}{(n-1)!} \sigma^n \qquad \text{Where } \sigma = \lambda/\mu$$

$$(ii)\ \lambda P_0 \frac{\partial}{\partial \lambda}\left[\frac{\rho^c}{c!}\ {}_1F_1(1;\delta+1;\beta\gamma)\right] =$$

$$\lambda P_0\ {}_1F_1(1;\delta+1;\beta\ \gamma) \frac{\partial}{\partial \lambda}\left[\frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c\right] + \lambda P_0 \frac{\rho^c}{c!} \frac{\partial}{\partial \lambda}\left[{}_1F_1(1;\delta+1;\beta\ \gamma)\right]$$

$$= \lambda P_0\ {}_1F_1(1;\delta+1;\beta\ \gamma) \frac{1}{c!} \frac{c\lambda^{c-1}}{\mu^c} + \lambda P_0 \frac{\rho^c}{c!} \frac{\partial}{\partial \lambda}\left[\sum_{n=0}^{\infty} \frac{(1)_n (\beta\gamma)^n}{(\delta+1)_n\ n!}\right]$$

$${}_1F_1(1;\delta+1;\beta\ \gamma) \frac{1}{(c-1)!}\left(\frac{\lambda}{\mu}\right)^c P_0 + \lambda P_0 \frac{\rho^c}{c!} \sum_{n=0}^{\infty} \frac{(1)_n\ \beta^n\ \lambda^{n-1} / \alpha^n}{(\delta+1)_n\ n!}$$

$${}_1F_1(1;\delta+1;\beta\ \gamma) \frac{\sigma^c}{(c-1)!} P_0 + P_0 \frac{\rho^c}{c!} \sum_{n=0}^{\infty} \frac{(1)_n n \lambda^n\ \beta^n / \alpha^n}{(\delta+1)_n\ n!}$$

$${}_1F_1(1;\delta+1;\beta\ \gamma) \frac{\sigma^c}{(c-1)!} P_0 + P_0 \frac{\rho^c}{c!} \sum_{n=0}^{\infty} \frac{(1)_n\ (\beta\gamma)^n}{(\delta+1)_n\ (n-1)!}$$

$${}_1F_1(1;\delta+1;\beta\ \gamma) \frac{\sigma^c}{(c-1)!} P_0 + P_0 \frac{\rho^c}{c!} \sum_{n=0}^{\infty} \frac{n!\ (\beta\gamma)^n}{(\delta+1)_n\ (n-1)!}$$

$${}_1F_1(1;\delta+1;\beta\ \gamma) \frac{\sigma^c}{(c-1)!} P_0 + P_0 \frac{\rho^c}{c!} \sum_{n=0}^{\infty} \frac{n\ (\beta\gamma)^n}{(\delta+1)_n}$$

$${}_1F_1(1;\delta+1;\beta\ \gamma) \frac{\sigma^c}{(c-1)!} P_0 + P_0 \frac{\rho^c}{c!}\left[\sum_{n=0}^{\infty} \frac{n\beta\gamma\ (\beta\gamma)^{n-1}}{(\delta+1)\ (\delta+2)_{n-1}}\right]$$

$${}_1F_1(1;\delta+1;\beta\ \gamma) \frac{\sigma^c}{(c-1)!} P_0 + P_0 \frac{\rho^c}{c!} \sum_{n=0}^{\infty} \frac{\beta\gamma n\ (\beta\gamma)^{n-1}}{(\delta+1)\ (\delta+2)_{n-1}}\Bigg]$$

$${}_1F_1(1;\delta+1;\beta\ \gamma) \frac{\sigma^c}{(c-1)!} P_0 + P_0 \frac{\rho^c}{c!} \sum_{n=1}^{\infty} \frac{\beta\gamma n\ (\beta\gamma)^{n-1}\ n!}{(\delta+1)\ (\delta+2)_{n-1}\ n!}\Bigg]$$

$$ {}_1F_1\,(1;\delta+1;\beta\,\gamma)\ \frac{\sigma^{c}}{(c-1)!}\ P_0 + P_0\,\frac{\rho^{c}}{c!}\sum_{n=1}^{\infty}\ \frac{(\beta\gamma)^{n-1}\ n!}{(\delta+1)\,(\delta+2)_{n-1}\ (n-1)!}\Bigg] $$

$$ {}_1F_1\,(1;\delta+1;\beta\,\gamma)\ \frac{\sigma^{c}}{(c-1)!}\ P_0 + P_0\,\frac{\rho^{c}}{c!}\sum_{n=0}^{\infty}\ \frac{(\beta\gamma)^{n}\ (1)_{n+1}}{(\delta+1)\,(\delta+2)_{n}\ n!}\Bigg] $$

$$ {}_1F_1\,(1;\delta+1;\beta\,\gamma)\ \frac{\sigma^{c}}{(c-1)!}\ P_0 + P_0\ \frac{\rho^{c}}{c!}\sum_{n=0}^{\infty}\ \frac{\beta\gamma\ (\beta\gamma)^{n}\ (2)_n}{(\delta+1)\,(\delta+2)_{n}\ n!}\Bigg] $$

$$ {}_1F_1\,(1;\delta+1;\beta\gamma)\ \frac{\sigma^{c}}{(c-1)!}\ P_0 + P_0\ \frac{\rho^{c}}{c!}\,\frac{\beta\gamma}{\delta+1}\ {}_1F_1\,(2;\delta+2;\beta\,\gamma) $$

$$ (III)\ \lambda\,P_0\ \frac{\partial}{\partial\lambda}\left[\frac{\rho^{c}}{c!}\ \frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\right.\ \ {}_2F_1\,(1;2;\delta+Y+2-c;\beta\gamma) = $$

$$ \lambda\,P_0\ \ {}_2F_1\,(1;2;\delta+Y+2-c;\beta\gamma)\ \frac{\partial}{\partial\lambda}\left[\frac{\rho^{c}\,(\beta\gamma)^{Y-c+1}}{c!\ (\delta+1)_{Y-c+1}}\right] + $$

$$ \lambda\,P_0\ \frac{\rho^{c}}{c!}\ \frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}}\ \frac{\partial}{\partial\lambda}\left[\ {}_2F_1\,(1;2;\delta+Y+2-c;\beta\gamma)\right] \quad (2.1.17) $$

$$ \text{(iii) (a)}\quad \lambda\,P_0\ \frac{\partial}{\partial\lambda}\left[\frac{\rho^{c}\,(\beta\gamma)^{Y-c+1}}{c!\ (\delta+1)_{Y-c+1}}\right] $$

$$ \lambda\,P_0\ \frac{\partial}{\partial\lambda}\left[\frac{\left(\lambda^{c}/\mu^{c}\right)\beta^{Y-c+1}\ \dfrac{\lambda^{Y-c+1}}{\alpha^{Y-c+1}}}{c!\ (\delta+1)_{Y-c+1}}\right] $$

$$ \lambda\,P_0\ \frac{\partial}{\partial\lambda}\ \frac{\left[\dfrac{1}{(\mu)^{c}}\ \beta^{Y-c+1}\ \dfrac{\lambda^{Y+1}}{\alpha^{Y-c+1}}\right]}{c!\ (\delta+1)_{Y-c+1}} $$

$$ \lambda\,P_0\ \left[\frac{\dfrac{1}{\mu^{c}}\ \beta^{Y-c+1}\ (Y+1)\ \dfrac{\lambda^{Y}}{\alpha^{Y-c+1}}}{c!\ (\delta+1)_{Y-c+1}}\right] $$

$$P_0 \left[\frac{\frac{1}{\mu^c} \beta^{Y\text{-}c+1} \; (Y+1)\lambda^{Y+1} \big/ \alpha^{Y\text{-}c+1}}{c! \; (\delta+1)_{Y\text{-}c+1}} \right]$$

$$P_0 \left[\frac{(Y+1)}{c! \; (\delta+1)_{Y\text{-}c+1}} \; \frac{\lambda^c}{\mu^c} \beta^{Y\text{-}c+1} \quad \lambda^{Y+1\text{-}c} \big/ \alpha^{Y+1\text{-}c} \right]$$

$$P_0 \left[\frac{(Y+1)}{c! \; (\delta+1)_{Y\text{-}c+1}} \; \rho^c \left(\beta\lambda \big/ \alpha \right)^{Y+1\text{-}c} \right]$$

$$P_0 \left[(Y+1) \; \frac{\rho^c}{c!} \; \frac{(\beta\gamma)^{Y\text{-}c+1}}{(\delta+1)_{Y\text{-}c+1}} \right]$$

$${}_2F_1\,(1,2;\delta+Y+2\text{-}c;\beta\gamma)\;\lambda P_0 \;\frac{\partial}{\partial\lambda} \left[\frac{\rho^c}{c!} \; \frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}} \right]$$

$${}_2F_1\,(1,2;\delta+Y+2\text{-}c;\beta\gamma)\;(Y+1)\;\frac{\rho^c}{c!} \; \frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+1}} \; P_o$$

$$(iii)\;(b)\;\lambda P_0 \;\frac{\partial}{\partial\lambda} \left[{}_2F_1\,(1,2;\delta+Y+2\text{-}c;\beta\gamma) \right]$$

$$= \lambda P_0 \;\frac{\partial}{\partial\lambda} \left[\sum_{n=0}^{\infty} \frac{(1)_n \; (2)_n \; \beta^n \big/ \alpha^n \; \lambda^n}{(\delta+Y+2\text{-}c)_n \; n!} \right]$$

$$\lambda P_0 \left[\sum_{n=0}^{\infty} \frac{(1)_n \; (2)_n \; n \; \lambda^{n\text{-}1} \; \beta^n \big/ \alpha^n}{n! \; (\delta+Y+2\text{-}c)_n} \right]$$

$$\lambda P_0 \left[\sum_{n=0}^{\infty} \frac{(1)_n \; (2)_n \; n \beta^n \big/ \alpha^n \; \lambda^n}{n! \; (\delta+Y+2\text{-}c)_n} \right]$$

$$P_0 \left[\sum_{n=0}^{\infty} \frac{(1)_n \; (2)_n \; n(\beta\gamma)^n}{n! \; (\delta+Y+2\text{-}c)_n} \right]$$

$$P_0 \left[\sum_{n=0}^{\infty} \frac{(1)_n \; (2)_n \; (\beta\gamma)^n}{(n\text{-}1)! \; (\delta+Y+2\text{-}c)_n} \right]$$

$$P_0 \sum_{n=0}^{\infty} \frac{1}{(\delta+Y+2-c)} \left[\frac{(1)_n\,(2)_n\,(\beta\gamma)^n}{(n-1)!\,(\delta+Y+3-c)_n} \right]$$

$$P_0 \sum_{n=1}^{\infty} \frac{1}{(\delta+Y+2-c)} .1.2.\beta\gamma \frac{(2)_{n-1}\,(3)_{n-1}\,(\beta Y)^{n-1}}{(n-1)!\,(\delta+Y+3-c)_{n-1}}$$

$$P_0 \frac{1 \quad 2\beta\gamma}{(\delta+Y+2-c)} \sum_{n=0}^{\infty} \frac{(2)_n\,(3)_n\,(\beta\gamma)^n}{n!\,(\delta+Y+3-c)_n}$$

$$P_0 \frac{2\beta\gamma}{(\delta+Y+2-c)} \; {}_2F_1\,(2,3;\delta+Y+3-c;\beta\gamma)$$

Put in (B) (2.1.17)

$$(Y+1)\; \frac{\rho^c\,(\beta\gamma)^{Y-c+1}}{c!\,(\delta+1)_{Y-c+1}}\; P_0\; {}_2F_1\,(1,2;\delta+Y+2-c;\beta\gamma)$$

$$+2\;\beta\gamma\; \frac{(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+2}} P_0\; {}_2F_1\,(2,3;\delta+Y+2-c;\beta\gamma)$$

Thus $L = \lambda\, P_0\, \dfrac{\partial}{\partial\lambda}\, P_0^{-1}$

$$= P_0 \left[\sum_{n=0}^{c-1} \frac{\sigma^n}{(n-1)!} + \frac{\sigma^c}{(c-1)!}\; {}_1F_1\;\;(1;\delta+1;\beta\gamma) + \right.$$

$$\frac{\rho^c}{c!} \frac{\beta\gamma}{(\delta+1)}\; {}_1F_1\;\;(2;\delta+2;\beta\gamma) +$$

$$\frac{(Y+1)\,\rho^c\;\;(\beta\gamma)^{Y-c+1}}{c!\qquad (\delta+1)_{Y-c+2}}\;\; {}_2F_1\,(1,2;\delta+Y+2-c;\beta\gamma)\; +$$

$$\frac{2\beta\gamma\,(\beta\gamma)^{Y-c+1}}{(\delta+1)_{Y-c+2}}\; {}_2F_1\,(2,3;\delta+Y+2-c;\beta\gamma) \qquad (2.1.18)$$

2.2 SPECIAL CASE

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad n=0 \qquad (2.2.1)$$

$$P_n'(t) = -(\lambda + n\mu) P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$$

$$1 \le n < c \quad (2.2.2)$$

$$P_c'(t) = -[\beta\lambda + c\mu] P_c(t) + \lambda P_{c-1}(t) + (c\mu + \alpha) P_{c+1}(t)$$

$$n = c \quad (2.2.3)$$

$$P_n'(t) = -[\beta\lambda + c\mu + (n-c)\alpha] P_n(t) + \beta\lambda P_{n-1}(t) + [c\mu + (n+1-c)\alpha] P_{n+1}(t)$$

$$c+1 \le n < Y \quad (2.2.4)$$

$$P_n'(t) = -[(n-Y+1)\beta\lambda + c\mu + (n-c)\alpha] P_n(t) + (n-Y)\beta\lambda P_{n-1}(t) +$$

$$[c\mu + (n+1-c)\alpha] P_{n+1}(t) \qquad Y+1 \le n < Y+K \qquad (2.2.5)$$

Let $\alpha = 0$, $\beta = 1$ then above equations are:

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad n=0 \qquad (2.2.6)$$

$$P_n'(t) = -(\lambda + n\mu) P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \quad 1 \le n < c \qquad (2.2.7)$$

$$P_c'(t) = -[\lambda + c\mu] P_c(t) + \lambda P_{c-1}(t) + c\mu P_{c+1}(t) \qquad n=c \qquad (2.2.8)$$

$$P_n'(t) = -(\lambda + c\mu) P_n(t) + \lambda P_{n-1}(t) + c\mu P_{n+1}(t) \quad c+1 \le n < Y \quad (2.2.9)$$

$$P_n'(t) = -[(n-Y+1)\lambda + c\mu] P_n(t) + (n-Y)\lambda P_{n-1}(t) + c\mu$$

$$P_{n+1}(t), \quad Y+1 \le n < Y+K \quad (2.2.10)$$

as t $\rightarrow\infty$ then the steady state probability difference equations are :-

$$-\lambda P_0 + \mu P_1 = 0 \qquad (2.2.11)$$

$$-(\lambda + n\mu) P_n(t) + \lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \quad (2.2.12)$$

$$-[\lambda + c\mu] P_c + \lambda P_{c-1} + c\mu P_{c+1} = 0 \qquad (2.2.13)$$

$$-(\lambda + c\mu) P_n + \lambda P_{n-1} + c\mu P_{n+1} = 0 \qquad (2.2.14)$$

$$-[(n-Y+1)\lambda + c\mu] P_n + (n-Y)\lambda P_{n-1} + c\mu P_{n+1} = 0 \quad (2.2.15)$$

From equation (2.2.11)

$$-\lambda P_0 + \mu P_1 = 0$$

$$\mu P_1 = \lambda P_0$$

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_1 = \rho P_0$$

From equation (2.2.12)

$$-(\lambda + n\mu) P_n + \lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \qquad 1 \le n < c$$

$$(n+1)\mu P_{n+1} = (\lambda + n\mu) P_n - \lambda P_{n-1}$$

Put n = 1 in above equation

$$2\mu P_2 = (\lambda + \mu) P_1 - \lambda P_0$$

$$2 P_2 = (\rho + 1) P_1 - \rho P_0$$

$$2 P_2 = (\rho + 1)\rho P_0 - \rho P_0$$

$$2 P_2 = \rho P_0(\rho + 1 - 1)$$

$$P_2 = \frac{\rho^2}{2!} P_0$$

Put n = 2 in above equation

$$3\mu P_3 = (\lambda + 2\mu) P_2 - \lambda P_1$$

$$3 P_3 = (\rho + 2) \frac{\rho^2}{2!} P_2 - \rho P_1$$

$$= \frac{\rho^2}{2!} P_0 [\rho+2-2]$$

$$P_3 = \frac{1}{3!} \rho^3 P_0$$

Similarly,

$$P_n = \frac{1}{n!} \rho^n P_0 \qquad 0 \leq n < c \quad (2.2.16)$$

From equation (2.2.13)

$$-(\lambda + c\mu) P_c + \lambda P_{c-1} + c\mu \, P_{c+1} = 0$$

$$c\mu \, P_{c+1} = (\lambda + c\mu) P_c - \lambda P_{c-1}$$

$$c \, P_{c+1} = (\rho + c) P_c - \rho \, P_{c-1} \qquad \text{where} \quad \rho = \lambda / \mu$$

$$c \, P_{c+1} = (\rho + c) \frac{1}{c!} \rho^c \, P_0 - \rho \frac{1}{(c-1)!} \rho^{c-1} P_0$$

$$c \, P_{c+1} = (\rho + c) \frac{1}{c!} \rho^c \, P_0 - \frac{\rho^c * c}{(c-1)! * c} P_0$$

$$= \frac{1}{c!} \rho^c \, P_0 [\rho + c - c]$$

$$P_{c+1} = \frac{1}{c!} \rho^c \, P_0 \left(\frac{\rho}{c} \right)$$

$$P_{c+1} = \frac{1}{c!} \rho^c \left(\frac{\rho}{c} \right) P_0$$

From equation (2.2.14)

$$-[\lambda + c\mu] P_n + \lambda \, P_{n-1} + c\mu \, P_{n+1} = 0$$

$$c\mu \, P_{n+1} = (\lambda + c\mu) P_n - \lambda P_{n-1}$$

$$c\,P_{n+1} = (\rho + c)\,P_n - \rho P_{n-1} \qquad 1 + c \le n < Y$$

bove equation

$$c\,P_{c+2} = (\rho + c)\,P_{c+1} - \rho\,P_c$$

$$= (\rho + c)\,\frac{1}{c!}\,\rho^c \left(\frac{\rho}{c}\right) P_0 - \frac{\rho}{c}\,\frac{1}{c!}\,\rho^c\,P_0\,c$$

$$c\,P_{c+2} = \frac{1}{c!}\,\rho^c P_0 \left(\frac{\rho}{c}\right) [\rho + c - c]$$

$$c\,P_{c+2} = \frac{1}{c!}\,\rho^c\,P_0 \left(\frac{\rho}{c}\right) \rho$$

$$P_{c+2} = \frac{1}{c!}\,\rho^c\,P_0 \left(\frac{\rho}{c}\right)^2$$

$$P_{c+2} = \frac{1}{c!}\,\rho^c \left(\frac{\rho}{c}\right)^2 P_0$$

Put n = c+2 in above equation

$$c\,P_{c+3} = (\rho + c)\,P_{c+2} - \rho P_{c+1}$$

$$= (\rho + c)\,\frac{1}{c!}\,\rho^c \left(\frac{\rho}{c}\right)^2 P_0 - \frac{\rho}{c}\,\frac{1}{c!}\,\rho^c \left(\frac{\rho}{c}\right) P_0\,c$$

$$= \frac{1}{c!}\,\rho^c \left(\frac{\rho}{c}\right)^2 P_0\,[\rho + c - c]$$

$$P_{c+3} = \frac{1}{c!}\,\rho^c \left(\frac{\rho}{c}\right)^2 P_0\,\frac{\rho}{c}$$

$$P_{c+3} = \frac{1}{c!}\,\rho^c \left(\frac{\rho}{c}\right)^3 P_0$$

Similarly

$$P_n = P_{c+(n-c)} = \frac{1}{c!}\,\rho^c \left(\frac{\rho}{c}\right)^{n-c} P_0 \qquad c < n < Y \text{ (2.2.17)}$$

From equation. (2.2.15)

$$-\left[(n-Y+1)\lambda + c\mu\right] P_n + (n-Y)\lambda P_{n-1} + c\mu\ P_{n+1} = 0$$

$$c\, P_{n+1} = \left[(n-Y+1)\rho + c\right] P_n - (n-Y)\rho P_{n-1}$$

Put $\mathbf{n} = Y+1$

$$c\, P_{Y+2} = \left[(Y+1-Y+1)\rho + c\right] P_{Y+1} - (Y+1-Y)\rho P_Y$$

$$c\, P_{Y+2} = \left[2\rho + c\right] P_{Y+1} - \rho P_Y$$

$$c\, P_{Y+2} = \left[2\rho + c\right] \frac{1}{c!}\rho^c \left(\frac{\rho}{c}\right)^{Y-c+1} P_0 - \frac{\rho}{c}\frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c} c\, P_0$$

$$= \frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c+1} P_0\left[2\rho + c - c\right]$$

$$c\, P_{Y+2} = \frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c+1} 2\rho\, P_0$$

$$P_{Y+2} = \frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c+1} 2\left(\frac{\rho}{c}\right) P_0$$

$$P_{Y+2} = \frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c} 2\left(\frac{\rho}{c}\right)^2 P_0$$

Put $\mathbf{n} = Y+2$

$$c\, P_{Y+3} = \left[(Y+2-Y+1)\rho + c\right] P_{Y+2} - (Y+2-Y)\rho P_{Y+2-1}$$

$$c\, P_{Y+3} = (3\rho + c) P_{Y+2} - 2\rho\, P_{Y+1}$$

$$c\, P_{Y+3} = (3\rho + c)\frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c} 2\left(\frac{\rho}{c}\right)^2 P_0\ -$$

$$2\rho\frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y+1-c} P_0$$

$$c\, P_{Y+3} = (3\rho + c)\frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c} 2\left(\frac{\rho}{c}\right)^2 P_0\ -$$

$$2\frac{\rho}{c}\frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c}\left(\frac{\rho}{c}\right) c$$

$$c\,P_{Y+3} = \frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c} 2\left(\frac{\rho}{c}\right)^2 P_0(3\rho + c - c)$$

$$P_{Y+3} = \frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c} 2.3\left(\frac{\rho}{c}\right)^3 P_0$$

$$P_{Y+3} = \frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c} 2.\ 3\left(\frac{\rho}{c}\right)^3 P_0$$

Similarly.

$$P_n = P_{Y+(n-Y)}$$

$$= \frac{1}{c!}\rho^c\left(\frac{\rho}{c}\right)^{Y-c} (n-Y)!\left(\frac{\rho}{c}\right)^{n-Y} P_0$$

$$P_n = \frac{1}{c!}\frac{\rho^Y}{c^{Y-c}}(n-Y)!\left(\frac{\rho}{c}\right)^{n-Y} P_0 \quad (2.2.18)$$

Thus P_n can be written as follows :-

$$P_n = \begin{cases} \dfrac{1}{n!}\rho^n P_0 & 0 < n < c \\ \dfrac{1}{c!}\rho^n\left(\dfrac{\rho}{c}\right)^{n-c} P_0 & c < n < Y \quad (2.2.19) \\ \dfrac{1}{c!}\dfrac{\rho^Y (n-Y)!}{c^{Y-c}}\left(\dfrac{\rho}{c}\right)^{n-Y} & Y+1 \le n < Y+k \end{cases}$$

To find P_0 the boundary condition $\sum_{n=0}^{\infty} P_n = 1$

$$P_0\left[\sum_{n=0}^{C-1}\frac{1}{n!}\rho^n + \frac{\rho^c}{c!}\sum_{n=c}^{Y}\left(\frac{\rho}{c}\right)^{n-c} + \frac{\rho^Y}{c!\,c^{Y-c}}\sum_{n=Y+1}^{\infty}(n-Y)\left(\frac{\rho}{c}\right)^{n-Y}\right] = 1$$

$$P_0^{-1} = \sum_{n=0}^{C-1}\frac{1}{n!}\rho^n + \frac{\rho^c}{c!}\sum_{n=c}^{Y}\left(\frac{\rho}{c}\right)^{n-c} + \frac{\rho^Y}{c!\,c^{Y-c}}\sum_{n=Y+1}^{\infty}(n-Y)!\left(\frac{\rho}{c}\right)^{n-Y} \quad (2.2.19)$$

(i). $\frac{\rho^{c}}{c!}\sum_{n=c}^{Y}\left(\frac{\rho}{c}\right)^{n-c}$

$$\frac{\rho^{c}}{c!}\sum_{n=0}^{Y-c}\left(\frac{\rho}{c}\right)^{n}\frac{n!}{n!}$$

$$\frac{\rho^{c}}{c!}\sum_{n=0}^{Y-c}\frac{(1)_{n}\left(\frac{\rho}{c}\right)^{n}}{n!}$$

$$\frac{\rho^{c}}{c!}\ {}_{1}F_{0}\ (1;-;\rho/c)$$

(ii). $\frac{\rho^{Y}}{c!\ c^{Y-c}}\sum_{n=Y+1}^{\infty}(n-Y)!\left(\frac{\rho}{c}\right)^{n-Y}$

$$\frac{\rho^{Y}}{c!\ c^{Y-c}}\sum_{n=0}^{\infty}(n-1)!\left(\frac{\rho}{c}\right)^{n+1}$$

$$\frac{\rho^{Y}}{c!\ c^{Y-c}}\sum_{n=0}^{\infty}\frac{\rho}{c}\frac{(2)_{n}\left(\frac{\rho}{c}\right)^{n}n!}{n!}$$

$$\frac{\rho^{Y+1}}{c!\ c^{Y-c+1}}\ {}_{2}F_{0}\left(1,2;-;\frac{\rho}{c}\right)$$

Hence

$$P^{-1}{}_{0}=\sum_{n=0}^{c-1}\frac{\rho^{n}}{n!}+\frac{\rho^{c}}{c!}\ {}_{1}F_{0}\ (1;-;\rho/c)+\frac{\rho^{Y+1}}{c!\ c^{Y-c+1}}\ {}_{2}F_{0}\left(1,2;-;\frac{\rho}{c}\right) \qquad (2.2.20)$$

and $L=\lambda P_{0}\ \frac{\partial}{\partial\lambda}\ P_{0}^{-1}$

$$L=\lambda P_{0}\ \frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{c-1}\frac{\rho^{n}}{n!}+\frac{\rho^{c}}{c!}\ {}_{1}F_{0}\ (1;-;\rho/c)+\frac{\rho^{Y+1}}{c!\ c^{Y-c+1}}\ {}_{2}F_{0}\left(1,2;-;\frac{\rho}{c}\right)\right] \qquad (2.2.21)$$

(i). $\lambda P_{0}\sum_{n=0}^{c-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^{n}$

$$\sum_{n=0}^{c-1}\frac{\rho^{n}}{(n-1)!}$$

$$\lambda P_0 \frac{\partial}{\partial \lambda}\left[\frac{\rho^c}{c!}\ 1\, F_0\,(1;-;\rho/c)\right]$$

$$\lambda P_0\ {}_1F_0\,(1;-;\rho/c)\ \frac{\partial}{\partial \lambda}\ \frac{\rho^c}{c!} + \lambda P_0\ \frac{\rho^c}{c!}\ \frac{\partial}{\partial \lambda}\ {}_1F_0\,(1;-;\rho/c)$$

$$\lambda P_0\ {}_1F_0\,(1;-;\rho/c)\ c\ \frac{\lambda^{c-1}}{\mu^c\ c!} + \lambda P_0\ \frac{\rho^c}{c!}\ \frac{\partial}{\partial \lambda}\ \sum_{n=0}^{\infty}\frac{(1)_n\ (\rho/c)^n}{n!}$$

$$P_0\ \frac{\rho^c}{(c-1)!}\ {}_1F_0\,(1;-;\rho/c)\ + \lambda P_0\ \frac{\rho^c}{c!}\ \sum_{n=0}^{\infty}(1)_n\ \frac{n\left(\dfrac{\lambda^{n-1}}{\mu^n}\right)\Big/c^n}{n!}$$

$$P_0\ \frac{\rho^c}{(c-1)!}\ {}_1F_0\,(1;-;\rho/c)\ +\ P_0\ \frac{\rho^c}{c!}\ \sum_{n=0}^{\infty}\frac{(1)_n\ (\rho/c)^n}{(n-1)!}$$

$$P_0\ \frac{\rho^c}{(c-1)!}\ {}_1F_0\,(1;-;\rho/c)\ +\ P_0\ \sum_{n=1}^{\infty}\frac{\rho^c}{c!}\frac{(2)_{n-1}\ (\rho/c)^{n-1}}{(n-1)!}$$

$$P_0\ \frac{\rho^c}{(c-1)!}\ {}_1F_0\,(1;-;\rho/c)\ +\ P_0\ \frac{\rho^c}{c!}\ \frac{\rho}{c}\sum_{n=0}^{\infty}\frac{(2)_n\ (\rho/c)^n}{n!}$$

$$P_0\ \frac{\rho^c}{(c-1)!}\ {}_1F_0\,(1;-;\rho/c)\ +\ P_0\ \frac{\rho^{c+1}}{c!\ c}\ {}_2F_0\,(1;-;\rho/c)$$

$$(iii).\ \lambda P_0\ \frac{\partial}{\partial \lambda}\left[\frac{\rho^{Y+1}}{c!\ c^{Y-c+1}}\ {}_2F_0(1,2;-;\rho/c)\right]$$

$$\lambda P_0\ {}_2F_0(1,2;-;\rho/c)\ \frac{\partial}{\partial \lambda}\left(\frac{\rho^{Y+1}}{c!\ c^{Y-c+1}}\right)+$$

$$\lambda P_0\ \frac{\rho^{Y+1}}{c!\ c^{Y-c+1}}\ \frac{\partial}{\partial \lambda}\left[\ {}_2F_0(1,2;-;\rho/c)\right]$$

$$\lambda P_0\ \frac{\partial}{\partial \lambda}\ \left(\rho^{Y+1}\right) = \lambda P_0\ \frac{1}{\mu^{Y+1}}\ (Y+1)\ \lambda^Y$$

$$= (Y+1)\frac{\lambda^{Y+1}}{\mu^{Y+1}}\ P_0$$

$$= (Y+1)\ \rho^{Y+1}\ P_0$$

$$\lambda P_0 \frac{\partial}{\partial \lambda}\left[\sum_{n=0}^{\infty} \frac{(2)_n\,(1)_n\,(\rho/c)^n}{n!}\right]$$

$$\lambda P_0 \left[\sum_{n=0}^{\infty} \frac{(2)_n\,(1)_n\,\frac{1}{c^n}\,\frac{1}{\mu^n}\,n\,\lambda^{n-1}}{n!}\right]$$

$$P_0 \left[\sum_{n=0}^{\infty} \frac{(2)_n\,(1)_n\,(\rho/c)^n}{(n-1)!}\right]$$

$$1.2 \sum_{n=1}^{\infty} (\rho/c)\,P_0\, \frac{(2)_{n-1}\,(3)_{n-1}\,(\rho/c)^{n-1}}{(n-1)!}$$

$$1.2\,(\rho/c)\,P_0 \sum_{n=0}^{\infty} \frac{(2)_n\,(3)_n\,(\rho/c)^{n-1}}{n!} = 1.2\,(\rho/c)\,P_0\ {}_2F_0\left(2,3;-;\frac{\rho}{c}\right)$$

$$L = P_0\left[\sum_{n=0}^{c-1} \frac{\rho^n}{(n-1)!} + \left(\frac{\rho^c}{(c-1)!}\right)\ {}_1F_0\,(1;-;\rho/c) + \frac{(Y+1)\rho^{Y+1}}{c!\,c^{Y-c+1}}\ {}_1F_1\,(1,2;-;\rho/c) + \right.$$

$$\frac{\rho^{c+1}}{c!\ c}\ {}_2F_0\,(2;-;\rho/c) +$$

$$\frac{(Y+1)\,\rho^{Y+1}}{c!\ c^{Y-c+1}}\ {}_2F_0\,(1,2;-;\rho/c) +$$

$$2\,\frac{\rho^{Y+2}}{c!c^{Y-c+2}}\ {}_2F_0\,(2,3;-;\rho/c) \qquad (2.2.22)$$

2.3 Case (II)

This case treats the system of Machine interference $M/M/C/K/\infty$ with balking reneging and spares $y < c$ and hence the birth - death coefficient are :-

$$\lambda_n = \begin{cases} \lambda & 0 \leq n < Y \\ (n-Y+1) & Y \leq n < c \\ (n-Y+1)\ \beta\lambda & c \leq n < Y+k \\ 0 & n \geq 0 \end{cases}$$

and

$$\mu_n \begin{cases} n\mu & 0 \leq n \leq c \\ c\mu + (n-c)\alpha & c+1 \leq n \leq Y+k \end{cases}$$

Then, as before, the probability differential - difference equations are :-

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad n=0 \qquad (2.3.1)$$

$$P_n'(t) = -(\lambda + n\mu)\, P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \qquad 1 \leq n \leq Y \qquad (2.3.2)$$

$$P_n'(t) = -\{ (n-Y+1)\lambda + n\mu] P_n(t) + (n-Y)\lambda P_{n-1}(t) + (n+1)\mu\, P_{n+1}(t)$$
$$Y+1 \leq n < c \qquad (2.3.3)$$

$$P_c'(t) = -\{ (c-Y+1)\beta\lambda + c\mu] P_c(t) + (c-Y)\lambda P_{c-1}(t) + (c\mu + \alpha)\, P_{c+1}(t) \quad n=c$$
$$Y+1 \leq n < c \qquad (2.3.4)$$

$$P_n'(t) = -\{ (n-Y+1)\beta\lambda + c\mu + (n-c)\alpha] P_n(t) + (n-Y)\beta\lambda P_{n-1}(t) + (c\mu + (n+1-c)\alpha)\, P_{n+1}(t) \quad c+1 \leq n < Y+K$$
$$(2.3.5)$$

There fore the steady state probability - difference equation are :-

$$-\lambda P_0 + \mu P_1 = 0 \qquad n = 0 \quad (2.3.6)$$

$$-(\lambda + n\mu)P_n + \lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \qquad 1 \le n < Y \quad (2.3.7)$$

$$-\left[(n - Y + 1)\lambda + n\mu\right] P_n + (n - Y)\lambda P_{n-1} + (n+1)\mu P_{n+1} = 0$$

$$Y + 1 \le n < c \quad (2.3.8)$$

$$-\left[(c - Y + 1)\beta\lambda + c\mu\right] P_c + (c - Y)\lambda P_{c-1} + (c\mu + \alpha) P_{c+1} = 0$$

$$n = c \quad (2.3.9)$$

$$-\left[(n - Y + 1)\beta\lambda + c\mu + (n-c)\alpha\right] P_n + (n - Y)\beta\lambda P_{n-1} +$$

$$\left[c\mu + (n+1-c)\alpha\right] P_{n+1} = 0 \qquad c + 1 \le n < Y + k \quad (2.3.10)$$

From equation (2.3.6)

$$-\lambda P_0 + \mu P_1 = 0$$

$$\mu P_1 = \lambda P_0$$

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_1 = \rho P_0$$

From equation (2.3.7)

$$-(\lambda + n\mu) P_n + \lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \qquad 1 \le n \le Y$$

$$(n+1)\mu P_{n+1} = (\lambda + n\mu) P_n - \lambda P_{n-1}$$

$$(n+1) P_{n+1} = (\rho + n) P_n - \rho P_{n-1}$$

Put n $= Y$

$$(Y+1)P_{Y+1} = (\rho + Y)P_Y - \rho P_{Y-1}$$

Put n = 1

$$2\ P_2 = (\rho+1)P_1 - \rho P_0$$

$$= (\rho+1)\rho P_0 - \rho P_0$$

$$2\ P_2 = \rho P_0\ (\rho+1-1)$$

$$P_2 = \frac{\rho^2}{2!} P_0$$

Put n = 2

$$3\ P_3 = (\rho+2)P_2 - \rho P_1$$

$$= (\rho+2)\ \frac{\rho^2}{2!} P_0 - \frac{\rho}{2}(\rho P_0) * 2$$

$$P_3 = \frac{\rho^2}{2!} P_0\ [\rho + 2 - 2]$$

$$P_3 = \frac{\rho^3}{3!} P_0$$

Put n = 3

$$4\ P_4 = (\rho+3)P_3 - \rho P_2$$

$$= (\rho+3)\frac{\rho^3}{3!} P_0 - \frac{\rho}{3}\left(\frac{\rho^2}{2!} P_0\right).3$$

$$= \frac{\rho^3}{3!} P_0\ [\rho + 3 - 3]$$

$$P_4 = \frac{\rho^4}{4!} P_0$$

Similary.

$$P_n = \frac{1}{n!}\rho^n P_0 \qquad 0 \le n < C \quad (2.3.11)$$

From equation (2.3.8)

$$-[(n-Y+1)\lambda + n\mu]P_n + (n-Y)\lambda P_{n-1} + (n+1)\mu P_{n+1} = 0$$

$$(n+1)\mu P_{n+1} = [(n-Y+1)\lambda + n\mu]P_n - (n-Y)\lambda P_{n-1}$$

Put $n = Y+1$

$$(Y+2)\mu P_{Y+2} = [(Y+1-Y+1)\lambda + (Y+1)\mu]P_{Y+1} - (Y+1-Y)\lambda P_Y$$

$$(Y+2)P_{Y+2} = \left[2\frac{\lambda}{\mu} + (Y+1)\frac{\mu}{\mu}\right]P_{Y+1} - \frac{\lambda}{\mu}P_Y$$

$$(Y+2)P_{Y+2} = [2\rho + Y+1]P_{Y+1} - \rho P_Y$$

$$(Y+2)P_{Y+2} = (2\rho + Y+1)\frac{\rho^{Y+1}}{(Y+1)!}P_0 - \rho\frac{1}{Y!}\rho^Y P_0$$

$$= \frac{\rho^Y}{Y!}P_0\left[\frac{2\rho + Y+1}{Y+1} - 1\right]\rho$$

$$= \frac{\rho^Y}{Y!}P_0\left[\frac{2\rho}{Y+1}\right]\rho$$

$$P_{Y+2} = \frac{\rho^Y}{Y!}\frac{2\rho^2}{(Y+1)(Y+2)}P_0$$

Put n $= Y+2$

$$(Y+3)\mu \, \mathrm{P}_{Y+3} = \left[(Y+2-Y+1)\lambda + (Y+2)\mu\right] \mathrm{P}_{Y+2} -$$

$$(Y+2-Y)\lambda \, \mathrm{P}_{Y+2-1}$$

$$(Y+3)\, \mathrm{P}_{Y+3} = \left[3\,\rho + (Y+2)\right] \mathrm{P}_{Y+2} - 2\,\rho\, \mathrm{P}_{Y+1}$$

$$(Y+3)\, \mathrm{P}_{Y+3} = \left[3\,\rho + (Y+2)\right] \frac{\rho^{Y}}{Y!} \; \frac{2\,\rho^{2}}{(Y+1)\,(Y+2)} P_0 -$$

$$2\,\rho \frac{\rho^{Y}}{Y!} \; \frac{\rho}{\mathrm{Y}+1} \; P_0$$

$$= \frac{\rho^{Y}}{\mathrm{Y!}} \; \frac{2\,\rho^{2}}{Y+1} P_0 \left[\frac{3\,\rho}{Y+2}\right]$$

$$\mathrm{P}_{Y+3} = \frac{\rho^{Y}\, 2.3\,\rho^{3}}{\mathrm{Y!}\,(Y+1)\,(Y+2)\,(Y+3)} P_0$$

Similarly,

$$P_n = P_{Y+(n-Y)} = \frac{\rho^{Y}\,(\mathrm{n-Y})!\,\rho^{\mathrm{n-Y}}}{\mathrm{Y!}\,(Y+1)_{n-Y}} P_0 \qquad Y \le \mathrm{n} \le \mathrm{c-1} \quad (2.3.12)$$

From equation (2.3.9)

$$-[(c-Y+1)\beta\lambda + c\mu]\,P_c + (c-Y)\lambda P_{c-1} + (c\mu+\alpha)P_{c+1} = 0 \qquad n=c$$

$$\frac{(c\mu+\alpha)}{\alpha}P_{c+1} = [(c-Y+1)\beta\gamma + \delta]\,P_c - (c-Y)\gamma P_{c-1}$$

$$(\delta+1)\,P_{c+1} = [(c-Y+1)\beta\gamma + \delta]\,P_c - (c-Y)\gamma\,P_{c-1}$$

$$P_{c+1} = \frac{(c-Y+1)}{\delta+1}\beta\gamma\;P_c + \frac{\delta}{\delta+1}P_c - \frac{(c-Y)\gamma}{\delta+1}\,P_{c-1} \qquad \text{(A)}$$

$$\frac{(c-Y)\gamma}{(\delta+1)}\;P_{c-1} = \frac{(c-Y)r}{(\delta+1)}\;\frac{\rho^{Y}\,(c-1-Y)!\,\rho^{c-1-Y}}{Y!\,(Y+1)_{c-1-Y}}\;P_o$$

$$= \frac{\rho^{Y}\,(c-Y)(c-1-Y)!\,\rho^{c-1-Y}\,.c.\rho\gamma}{Y!\;c\,(Y+1)_{c-1-Y}\;\rho\;(\delta+1)}\;P_o$$

$$= \frac{\rho^{Y}\,(c-Y)!\rho^{c-Y}}{Y!\;\;(Y+1)_{c-Y}}\;P_o\;\frac{c\,\gamma}{\rho(Y+1)}$$

$$= P_c\;\frac{c\;\lambda/\alpha}{\lambda/\mu\,(\delta+1)}$$

$$= \frac{\delta}{\delta+1}\;P_c$$

Put in (A) we get.

$$P_{c+1} = \frac{(c-Y+1)}{\delta+1}\;\beta\gamma\;P_c + \frac{\delta}{\delta+1}\;P_c - \frac{\delta}{\delta+1}\;P_c$$

$$P_{c+1} = \frac{(c-Y+1)}{\delta+1}\;\beta\gamma P_c$$

From equation (2.3.10)

$$-\left[(n-Y+1)\beta\lambda + c\mu + (n-c)\alpha\right] P_n + (n-Y)\beta\lambda P_{n-1} +$$

$$\left[c\mu + (n+1-c)\alpha\right] P_{n+1} = 0 \quad c+1 \le n < Y+k$$

$$\left[c\mu + (n+1-c)\alpha\right] P_{n+1} = \left[(n-Y+1)\beta\lambda + c\mu + (n-c)\alpha\right] P_n - (n-Y)\beta\lambda P_{n-1}$$

$$\left[\delta + n + 1 - c\right] P_{n+1} = \left[(n-Y+1)\beta\gamma + \delta + (n-c)\right] P_n -$$

$$(n-Y)\beta\gamma \; P_{n-1}$$

Put n = c + 1

$$\left[\delta + c + 1 + 1 - c\right] P_{c+2} = \left[(c+1-Y+1)\beta\gamma + \delta + c + 1 - c\right] P_{c+1} -$$

$$(c+1-y)\beta\gamma \; P_{c+1-1}$$

$$(\delta+2)\; P_{c+2} = \left[(c-Y+2)\beta\gamma + \delta + 1\right] P_{c+1} - (c-Y+1)\beta\gamma \; P_c$$

$$= \left[(c-Y+2)\beta\gamma + \delta + 1\right] \frac{(c-Y+1)\beta\gamma}{\delta+1} \; P_c - (c-Y+1)\beta\gamma \; P_c$$

$$= (c-Y+1)\beta\gamma \;\; P_c\left[\frac{(c-Y+2)\beta\gamma \;\; + \delta + 1}{\delta+1} \; - 1\right]$$

$$= (c-Y+1)\beta\gamma \; P_c \frac{(c-Y+2)\beta\gamma}{(\delta+1)}$$

$$P_{c+2} = \frac{(c-Y+1)\;(c-Y+2)\;(\beta\gamma)^2}{(\delta+1)\;(\delta+2)} \; P_c$$

$$P_{c+2} = \frac{(c-Y+1)\;(c-Y+2)}{(\delta+1)\;(\delta+2)} \; \frac{\rho^{Y}\,(c-Y)!\,\rho^{c-Y}}{Y!\;(Y+1)_{c-Y}} \; P_0$$

Put n = c + 2

$$(\delta+3)P_{c+3} = \left[(c+2-Y+1)\beta\gamma \ +\delta + \mathrm{c} + 2 - \mathrm{c}\right] \mathrm{P}_{c+2}$$

$$-(c+2-Y)\beta\gamma \quad \mathrm{P}_{c+1}$$

$$= \left[(c-Y+3)\beta\gamma + \delta + 2\right] \mathrm{P}_{c+2} - \left[c - Y + 2\right] \beta\gamma \quad \mathrm{P}_{c+1}$$

$$= \left[(c-Y+3)\beta\gamma \ +\delta + 2\right] \frac{(c-Y+1)\,(c-Y+2)\,(\beta\gamma)^2}{(\delta+1)\,(\delta+2)} \mathrm{P}_c -$$

$$(c-Y+2)\beta\gamma \quad \frac{(c-Y+1)\,\beta\gamma}{\delta+1} \mathrm{P}_c$$

$$= \frac{(c-Y+1)\,(c-Y+2)\,(\beta\gamma)^2}{(\delta+1)} \mathrm{P}_c \left[\frac{(c-Y+3)\beta\gamma + \delta + 2}{\delta+2} - 1\right]$$

$$= \frac{(c-Y+1)\,(c-Y+2)\,(\beta\gamma)^2}{(\delta+1)} \mathrm{P}_c \ \frac{(c-Y+3)\,\beta\gamma}{(\delta+2)}$$

$$P_{c+3} = \frac{(c-Y+1)\,(c-Y+2)\,(c-Y+3)\,(\beta\gamma)^3}{(\delta+1)\,(\delta+2)\,(\delta+3)} \mathrm{P}_c$$

$$P_{c+3} = \frac{(c-Y+1)\,(c-Y+2)\,(c-Y+3)\,(\beta\gamma)^3}{(\delta+1)\,(\delta+2)\,(\delta+3)} \ \frac{\rho^Y\,(\mathrm{c-Y})!\,\rho^{\mathrm{c-Y}}}{\mathrm{Y}!\,(Y+1)_{c-Y}} \mathrm{P}_0$$

Similarly,

$$P_n = P_{c+(n-c)} =$$

$$\frac{\rho^Y\,(\mathrm{c-Y})!\,\rho^{\mathrm{c-Y}}}{Y!\,(Y+1)_{c-Y}} \ \frac{(\mathrm{c-Y+1})_{\mathrm{n-c}}\,(\beta\gamma)^{n-c}}{(\delta+1)_{n-c}} \mathrm{P}_0 \qquad \mathrm{c} \le \mathrm{n} \le Y + \mathrm{k} \quad (2.3.13)$$

The empty system probability P_0, can be found from the

boundary condition: $\sum_{n=0}^{\infty} P_n = 1$

$$P_0^{-1} = \sum_{n=0}^{Y-1} \frac{1}{n!} \rho^n + \frac{\rho^Y}{Y!} \sum_{n=Y}^{c-1} \frac{(n-Y)!\, \rho^{n-Y}}{(Y+1)_{n-Y}} +$$

$$\sum_{n=c}^{\infty} \frac{\rho^Y (c-Y)!\, \rho^{c-Y}}{Y!\,(Y+1)_{c-Y}} \frac{(c-Y+1)_{n-c}\,(\beta\gamma)^{n-c}}{(\delta+1)_{n-c}} \tag{2.3.14}$$

$$(I)\ \frac{\rho^Y}{Y!} \sum_{n=Y}^{c-1} \frac{(n-Y)!\, \rho^{n-Y}}{(Y+1)_{n-Y}} \qquad n \rightarrow n+Y$$

$$\frac{\rho^Y}{Y!} \sum_{n=0}^{c-1-Y} \frac{n!\, \rho^n\, n!}{(Y+1)_n\, n!}$$

$$\frac{\rho^Y}{Y!} \sum_{n=0}^{c-1-Y} \frac{(1)_n\,(1)_n\, \rho^n}{(Y+1)_n\, n!}$$

$$\frac{\rho^Y}{Y!}\ {}_2F_1\,(1,1;Y+1;\rho)$$

$$(II) \sum_{n=c}^{\infty} \frac{\rho^Y (c-Y)!\, \rho^{c-Y}}{Y!\,(Y+1)_{c-Y}} \frac{(c-Y+1)_{n-c}\,(\beta\gamma)^{n-c}}{(Y+1)_{n-c}}$$

$$\frac{\rho^c\,(c-Y)!}{Y!\,(Y+1)_{c-Y}} \sum_{n=0}^{\infty} \frac{(c-Y+1)_n\,(\beta\gamma)^n}{(\delta+1)_n} \qquad n \rightarrow n+c$$

$$\frac{\rho^c}{Y!} \frac{(c-Y)!}{(Y+1)_{c-Y}} \sum_{n=0}^{\infty} \frac{(c-Y+1)_n\,(\beta\gamma)^n\,(1)_n}{(Y+1)_n\, n!}$$

$$\frac{\rho^c}{Y!} \frac{(c-Y)!}{(Y+1)_{c-Y}}\ {}_2F_1\,(1,c-Y+1;\delta+1;\beta\gamma)$$

$$P_0^{-1} = \sum_{n=0}^{Y-1} \frac{1}{n!}\, \rho^{n} + \frac{\rho^{Y}}{Y!}\ {}_2F_1\,(1,1;Y+1;\rho)\ +$$

$$\frac{\rho^{c}\,(c-Y)!}{Y!\,(Y+1)_{c-Y}}\ {}_2F_1\,(1,c-Y+1;\delta+1;\beta\gamma) \quad (2.3.15)$$

$$L = \lambda\, P_0\ \frac{\partial}{\partial\lambda}\ P_0^{-1}$$

$$= \lambda\, P_0\ \frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{Y-1} \frac{1}{n!}\, \rho^{n} + \frac{\rho^{Y}}{Y!}\ {}_2F_1\,(1,1;Y+1;\rho) + \right.$$

$$\frac{\rho^{c}}{Y!}\,\frac{(c-Y)!}{(Y+1)_{c-Y}}\ {}_2F_1\,(1,c-Y+1;\delta+1;\beta\gamma)$$

$$(I)\ \lambda\, P_0\ \frac{\partial}{\partial\lambda}\ \sum_{n=0}^{Y-1} \frac{1}{n!}\, \rho^{n}$$

$$\lambda\, P_0\ \frac{\partial}{\partial\lambda}\ \sum_{n=0}^{Y-1} \frac{1}{n!}\ \frac{\lambda^{n}}{\mu^{n}}$$

$$\lambda\, P_0\ \frac{\partial}{\partial\lambda}\ \sum_{n=0}^{Y-1} \frac{1}{n!}\ n\ \frac{\lambda^{n-1}}{\mu^{n}}$$

$$= P_0\ \sum_{n=0}^{Y-1} \frac{\rho^{n}}{(n-1)!}$$

$$(II)\ \lambda\, P_0\ \frac{\partial}{\partial\lambda}\left[\frac{\rho^{Y}}{Y!}\ {}_2F_1\,(1,1;Y+1;\rho)\right] =$$

$$\lambda\, P_0\,\frac{\partial}{\partial\lambda}\left[\frac{\rho^{Y}}{Y!}\right]\ {}_2F_1\,(1,1;Y+1;\rho) + \lambda\, P_0\ \frac{\rho^{Y}}{Y!}\ \frac{\partial}{\partial\lambda}\ [\ {}_2F_1\,(1,1;Y+1;\rho)]$$

$$(II)a\ \lambda\, P_0\ \frac{\partial}{\partial\lambda}\left[\frac{\rho^{Y}}{Y!}\right]\ {}_2F_1\,(1,1;Y+1;\rho)$$

$$\lambda\, P_0\ \ Y\frac{\lambda^{Y-1}}{\mu^{Y}\ Y!}\Bigg]\ {}_2F_1\,(1,1;Y+1;\rho)$$

$$P_0\left[\frac{\rho^{Y}}{(Y-1)\ !}\right]\ {}_2F_1\,(1,1;Y+1;\rho)$$

$$(II)\ \mathrm{b}\ \lambda\, P_0\ \frac{\rho^{Y}}{Y!}\ \frac{\partial}{\partial\lambda}\left[2\,F_1\,(1,1;Y+1;\rho)\right]$$

$$=\lambda\, P_0\,\frac{\rho^{Y}}{Y!}\left[\sum_{n=0}^{\infty}\frac{(1)_n\,(1)_n\ n\ \lambda^{n-1}\big/\mu^{n}}{(Y+1)_n\ n!}\right]$$

$$=P_0\,\frac{\rho^{Y}}{Y!}\sum_{n=0}^{\infty}\frac{(1)_n\ (1)_n\ \rho^{n}}{(Y+1)_n(n-1)!}$$

$$=P_0\,\frac{\rho^{Y}}{Y!}\sum_{n=1}^{\infty}\frac{1.\ 1\ \rho(2)_{n-1}(2)_{n-1}\ \rho^{n-1}}{(Y+1)\ (Y+2)_{n-1}\ (n-1)!}$$

$$=P_0\,\frac{\rho^{Y+1}}{(Y+1)\ !}\sum_{n=0}^{\infty}\frac{(2)_n\ (2)_n\ \rho^{n}}{(Y+2)_n\ n!}$$

$$=P_0\,\frac{\rho^{Y+1}}{(Y+1)\ !}\ \ {}_2F_1\,(2,2;Y+2;\rho)$$

$$(III)\ \lambda\, P_0\ \frac{\partial}{\partial\lambda}\left[\frac{\rho^{c}}{Y!}\ \frac{(c-Y)!}{(Y+1)_{c-Y}}\ \ {}_2F_1\,(1,c-Y+1;\delta+1\beta\gamma)\right]$$

$$=\ {}_2F_1\,(1,c-Y+1;\delta+1;\beta\gamma)\ \lambda\, P_0\ \frac{\partial}{\partial\lambda}\left[\frac{\rho^{c}\,(c-Y)!}{Y!\,(Y+1)_{c-Y}}\right]$$

$$+\,\frac{\rho^{c}}{Y!}\ \frac{(c-Y)!}{(Y+1)_{c-Y}}\ \lambda\, P_0\ \frac{\partial}{\partial\lambda}\left[\ {}_2F_1\,(1,c-Y+1;\delta+1;\beta\gamma)\right]$$

$$(III)\mathrm{a}\ \lambda\, P_0\ \frac{\partial}{\partial\lambda}\left[\frac{\rho^{c}\,(c-Y)!}{Y!\,(Y+1)_{c-Y}}\right]=\lambda\, P_0\left[\frac{c\,\lambda^{c-1}}{\dfrac{\mu^{c}}{Y!}}\ \ \frac{(c-Y)!}{(Y+1)_{c-Y}}\right]$$

$$=P_0\left[c\,\frac{\rho^{c}}{Y!}\ \ \frac{(c-Y)!}{(Y+1)_{c-Y}}\right]$$

$$(\mathrm{III})\ \mathrm{b}\ \lambda P_0 \frac{\partial}{\partial\lambda}\ {}_2F_1\left(1, c-Y+1; \delta+1; \beta\gamma\right)$$

$$\lambda P_0 \frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{\infty}\frac{(1)_n\ (c-Y+1)_n\ (\beta\gamma)^n}{(\delta+1)_n\ n!}\right]$$

$$P_0 \sum_{n=0}^{\infty}\frac{(1)_n\ (c-Y+1)_n}{(\delta+1)_n}\ \frac{\beta^n\lambda^{n-1}n/\alpha^n}{n!}$$

$$P_0 \sum_{n=0}^{\infty}\frac{(1)_n\ (c-Y+1)_n}{(\delta+1)_n}\ \frac{(\beta\gamma)^n}{(n-1)!}$$

$$P_0 \sum_{n=1}^{\infty}\frac{1\ (c-Y+1)\ \beta\gamma}{(\delta+1)}\ \frac{(2)_{n-1}\ (c-Y+2)_{n-1}\ (\beta\gamma)^{n-1}}{(\delta+2)_{n-1}\ (n-1)!}$$

$$P_0 \frac{\beta\gamma\ (c-Y+1)}{(\delta+1)} \sum_{n=0}^{\infty}\frac{(2)_n\ (c-Y+2)_n\ (\beta\gamma)^n}{(\delta+2)_n\ n!}$$

$$P_0 \frac{\beta\gamma(c-Y+1)}{(\delta+1)}\ {}_2F_1\left(2, c-Y+2; \delta+2; \beta\gamma\right)$$

(III)

$$\lambda P_0 \frac{\partial}{\partial\lambda}\left[\frac{\rho^c}{Y!}\ \frac{(c-Y)}{(Y+1)_{c-Y}}\ {}_2F_1\left(1, c-Y+1; \delta+1; \beta\gamma\right)\right]$$

$$= {}_2F_1\left(1, c-Y+1; \delta+1; \beta\gamma\right) c\ \frac{\rho^c}{Y!}\ \frac{(c-Y)!}{(Y+1)_{c-Y}}\ P_0 +$$

$$\frac{\rho^c}{Y!}\ \frac{(c-Y)!}{(Y+1)_{c-Y}}\ \frac{\beta\gamma\ (c-Y+1)}{\delta+1}\ {}_2F_1\left(2, c-Y+1; \delta+2; \beta\gamma\right) P_0$$

Hence.

$$L = P_0\left[\sum_{n=1}^{Y-1}\frac{\rho^n}{(n-1)!} + \frac{\rho^Y}{(Y-1)}\ {}_2F_1\left(1,1; Y+1; \rho\right) +\right.$$

$$\frac{\rho^{Y+1}}{(Y+1)!}\ {}_2F_1\left(2,2; Y+2; \rho\right) + \frac{c\,\rho^c\,(c-Y)!}{Y!\ (Y+1)_{c-Y}}\ {}_2F_1\left(1, c-Y+1; \delta+1; \beta\gamma\right)$$

$$\left.+\frac{\beta\gamma(c-Y+1)!\ \rho^c}{Y!\ (Y+1)_{c-Y}\ (\delta+1)}\ {}_2F_1\left(2, c-Y+2; \delta+2; \beta\gamma\right)\right] \qquad (2.3.16)$$

2.4 SPCEIAL CASE

Let $\alpha = 0$ and $\beta = 1$ then $M/M/C/K/\infty$ with spares only

we have.

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad n = 0 \quad (2.4.1)$$

$$P_n'(t) = -(\lambda + n\mu) P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$$
$$1 \le n < Y \qquad (2.4.2)$$

$$P_n'(t) = -[(n-Y+1)\lambda + n\mu] P_n(t) + (n-Y)\lambda P_{n-1}(t) +$$
$$(n+1)\mu P_{n+1}(t) \quad Y+1 \le n < c \qquad (2.4.3)$$

$$P_c'(t) = -[(c-Y+1)\lambda + c\mu] P_c(t) + (c-Y)\lambda P_{c-1}(t) +$$
$$c\,\mu + P_{c+1}(t) \quad n = c \qquad (2.4.4)$$

$$P_n'(t) = -[(n-Y+1)\lambda + c\mu] P_n(t) + (n-Y)\lambda P_{n-1}(t)$$
$$+\ c\,\mu\, P_{n+1}(t) \quad c+1 \le n < Y+k \qquad (2.4.5)$$

As t - ∞ then the steady state probability difference equation are.

$$-\lambda P_0 + \mu P_1 = 0 \qquad n = 0 \ (2.4.6)$$

$$-(\lambda + n\mu) P_n + \lambda P_{n-1} + (n+1)\mu P_{n+1} = 0$$
$$1 \le n < Y \quad (2.4.7)$$

$$-[(n-Y+1)\lambda + n\mu] P_n + (n-Y)\lambda P_{n-1} +$$
$$(n+1)\mu P_{n+1} = 0 \ \ Y+1 \le n < c \ (2.4.8)$$

$$-[(c-Y+1)\lambda + c\mu] P_c + (c-Y)\lambda P_{c-1} +$$
$$c\,\mu P_{c+1} = 0 \ \ n = c \ (2.4.9)$$

$$-[(n-Y+1)\lambda + c\mu] P_n + (n-Y)\lambda P_{n-1} +$$
$$c\,\mu P_{n+1} = 0 \ \ c+1 \le n < Y+k \ (2.4.10)$$

From equation (2.4.6)

$$-\lambda P_0 + \mu P_1 = 0$$

$$\mu P_1 = \lambda P_0$$

$$P_1 = \lambda/\mu \; P_0$$

$$P_1 = \rho \, P_0$$

From equation (2.4.7)

$$-(\lambda + n\mu)P_n + \lambda P_{n-1} + (n+1)\mu P_{n+1} = 0 \qquad 1 \le n < Y$$

$$(n+1)\mu P_{n+1} = (\lambda + n\mu)P_n - \lambda P_{n-1}$$

Put n = 1

$$2\,\mu P_2 = (\lambda + \mu)P_1 - \lambda P_0$$

$$2\; P_2 = (\rho + 1)P_1 - \rho P_0$$

$$= (\rho + 1)\rho P_0 - \rho P_0$$

$$2\; P_2 = \rho P_0\,(\rho + 1 - 1)$$

$$P_2 = \frac{\rho^2}{2!} P_0$$

Put n = 2

$$3\mu\; P_3 = (\lambda + 2\mu)P_2 - \lambda P_1$$

$$= (\lambda + 2\mu)\frac{\rho^2}{2!} P_0 - \lambda\;(\rho P_0)$$

$$3\,P_3 = (\rho + 2)\;\frac{\rho^2}{2!} P_0 - \frac{\rho^2}{2!} P_0\, 2$$

$$= \frac{\rho^2}{2!} P_0\,(\rho + 2 - 2)$$

$$P_3 = \frac{\rho^3}{3!} P_0$$

Similarly,

$$P_n = \frac{1}{n!}\rho^n P_0, \qquad 0 \le n < Y - 1 \qquad (2.4.11)$$

From equation (2.4.8)

$$-[(n-Y+1)\lambda + n\mu]\, P_n + (n-Y)\lambda P_{n-1} + (n+1)\mu P_{n+1} = 0$$

$$Y+1 \le n < c$$

$$(n+1)\mu P_{n+1} = [(n-Y+1)\,\lambda + n\,\mu]\, P_n - (n-Y)\lambda P_{n-1}$$

Put n = $Y+1$

$$(Y+2)\mu P_{Y+2} = [(Y+1-Y+1)\lambda + (Y+1)\mu)]\, P_{Y+1} - (Y+1-Y)\lambda P_Y$$

$$(Y+2)\mu P_{Y+2} = [2\,\lambda + (Y+1)]\, P_{Y+1} - \lambda P_Y$$

$$(Y+2)\, P_{Y+2} = [2\,\rho + (Y+1)]\, P_{Y+1} - \rho P_Y$$

$$= [2\,\rho + (Y+1)]\, \frac{\rho^{Y+1}}{(Y+1)!} P_0 - \rho \frac{\rho^Y}{(Y+1)\,Y!} P_0\,(Y+1)$$

$$= \frac{\rho^{Y+1}}{(Y+1)!} P_0\, [2\,\rho + (Y+1) - (Y+1)]$$

$$= \frac{\rho^{Y+1}}{(Y+1)!} P_0 .2\,\rho$$

$$P_{Y+2} = \frac{\rho^{Y+1}}{(Y+1)!}\,\frac{2\,\rho}{(Y+2)} P_0$$

$$P_{Y+2} = \frac{\rho^Y}{Y!}\,\frac{2\,\rho^2}{(Y+1)\,(Y+2)} P_0$$

Put $n = Y+2$

$$(Y+3)\,\mu P_{Y+3} = \left[(Y+2-Y+1)\lambda + (Y+2)\mu\right] P_{Y+2} - (Y+2-Y)\,\lambda P_{Y+1}$$

$$(Y+3)\,\mu P_{Y+3} = \left[3\,\lambda + (Y+2)\mu\right] P_{Y+2} - 2\lambda P_{Y+1}$$

$$(Y+3)\;P_{Y+3} = \left[3\,\rho + (Y+2)\right] P_{Y+2} - 2\rho\;P_{Y+1}$$

$$= \left[3\,\rho + (Y+2)\right] \frac{\rho^{Y}}{Y!}\;\frac{2\,\rho^{2}}{(Y+1)\,(Y+2)}\,P_0 - 2\,\rho\,\frac{\rho^{Y+1}}{(Y+1)\,!}\,P_0$$

$$= \left[3\,\rho + (Y+2)\right] \frac{\rho^{Y}}{Y!}\;\frac{2\,\rho^{2}}{(Y+1)\,(Y+2)}\,P_0 - \frac{2\,\rho^{2}\,\rho^{Y}}{Y!\,(Y+1)}\,P_0$$

$$= \frac{\rho^{Y}}{Y!}\;\frac{2\,\rho^{2}}{Y+1}\;P_0\left[\frac{3\,\rho + (Y+2)}{Y+2}\;-1\right]$$

$$= \frac{\rho^{Y}}{Y!}\;\frac{2\,\rho^{2}}{Y+1}\;\frac{3\rho}{Y+2}\;P_0$$

$$P_{Y+3} = \frac{\rho^{Y}}{Y!}\;\frac{2.3\quad \rho^{3}}{(Y+1)\,(Y+2)\,(Y+3)}\;P_0$$

Similarly,

$$P_n = P_{Y+(n-Y)} = \frac{\rho^{Y}\,(n-Y)!\,\rho^{n-Y}}{Y!\,(Y+1)_{n-Y}}\;P_0 \qquad Y \le n < c-1 \quad (2.4.12)$$

From equation (2.4.9)

$$-\left[(c-Y+1)\lambda + c\mu\right] P_c + (c-Y)\,\lambda P_{c-1} + c\mu P_{c+1} = 0$$

$$c\mu P_{c+1} = \left[(c-Y+1)\lambda + c\mu\right] P_c - (c-Y)\ \lambda P_{c-1}$$

$$c\, P_{c+1} = \left[(c-Y+1)\rho + c\right] P_c - (c-Y)\ \rho P_{c-1}$$

$$= \left[(c-Y+1)\rho + c\right] \frac{\rho^{Y}\ (c-Y)!\ \rho^{c-Y}}{Y!\ (Y+1)_{c-Y}}\ P_0$$

$$-(c-Y)\ \rho\ \frac{c\,\rho^{Y}\ (c-1-Y)!\ \rho^{c-1-Y}}{c\,Y!\ (Y+1)_{c-1-Y}}\ P_0$$

$$= \frac{\rho^{Y}\ (c-Y)!\ \rho^{c-Y}}{Y!\ (Y+1)_{c-Y}} P_0 \left[(c-Y+1)\ \rho + c - c\right]$$

$$c\, P_{c+1} = \frac{\rho^{Y}\ (c-Y)!\ \rho^{c-Y}}{Y!\ (Y+1)_{c-Y}} (c-Y+1)\ \rho P_0$$

$$P_{c+1} = \frac{\rho^{Y}}{Y!}\ \frac{(c-Y+1)!\ \rho^{c-Y+1}}{c\,(Y+1)_{c-Y}}\ P_0 = \frac{\rho^{Y}\ (c-Y+1)\ (c-Y)!\ \ \rho^{c-Y+1}}{Y!\ c\ (Y+1)_{c-Y}} P_0$$

From equation (2.4.10)

$$-\left[(n-Y+1)!\lambda + c\mu\right] P_n + (n-Y)\ \ \lambda P_{n-1} + c\mu P_{n+1} = 0$$

$$c\mu P_{n+1} = \left[(n-Y+1)\lambda + c\mu\right] P_n - (n-Y)\ \ \lambda P_{n-1}$$

$$c\, P_{n+1} = \left[(n-Y+1)\rho + c\right] P_n - (n-Y)\ \ \rho P_{n-1}$$

Put n = c + 1

$$c\,P_{c+2} = \left[(c+1-Y+1)\rho + c\right] P_{c+1} - (c+1-Y)\ \rho P_{c+1-1}$$

$$= \left[(c-Y+2)\rho + c\right] P_{c+1} - (c+1-Y)!\ \rho P_c$$

$$= \left[(c-Y+2)\rho + c\right] \frac{\rho^{Y}\ (c-Y+1)!\ \rho^{c-Y+1}}{Y!\ c\ (Y+1)_{c-Y}}\ P_0 -$$

$$\frac{c}{c}\ (c-Y+1)\rho\ \frac{\rho^{Y}\ (c-Y)!\ \rho^{c-Y}}{Y!\ c\ (Y+1)_{c-Y}}\ P_0$$

$$= \frac{\rho^{Y}\ (c-Y+1)!\ \rho^{c-Y+1}}{Y!\ c\ (Y+1)_{c-Y}}\ P_0\ \left[(c-Y+2)\rho + c - c\right]$$

$$c\,P_{c+2} = \frac{\rho^{Y}}{Y!}\ \frac{(c-Y+1)!\ \rho^{c-Y+1}}{c\ (Y+1)_{c-Y}}\ P_0\ (c-Y+2)\rho$$

$$P_{c+2} = \frac{\rho^{Y}\ (c-Y+1)!\ \rho^{c-Y+1}\ (c-Y+2)\ \rho}{Y!\ c^2\ (Y+1)_{c-Y}}\ P_0$$

$$P_{c+2} = \frac{\rho^{Y}\ (c-Y+2)!\ \rho^{c-Y+2}}{Y!\ c^2\ (Y+1)_{c-Y}}\ P_0 = \frac{\rho^{Y}\ (c-Y+2)\ (c-Y+1)\ (c-Y)!\ \rho^{c-Y+2}}{Y!\ c^2\ (Y+1)_{c-Y}}\ P_0$$

$$= \frac{(c-Y+1)\ (c-Y+2)}{(c\mu)^2} \cdot \frac{\lambda^2\ \rho^{Y}}{Y!}\ \frac{(c-Y)!\ \rho^{c-Y}}{(Y+1)_{c-Y}}\ P_0$$

$$= \frac{(c-Y+1)\ (c-Y+2)}{(c\mu)^2}\ P_c$$

Put n = c + 2

$$c\,P_{c+3} = \left[(c+2-Y+1)\ \rho + c\ \right] P_{c+2} - (c+2-Y)\ \rho P_{c+1}$$

$$= \left[(c-Y+3)\ \rho + c\ \right] \frac{\rho^{Y}\ (c-Y+2)!\ \rho^{c-Y+2}}{Y!\ c^2\ (Y+1)_{c-Y}}\ P_0 -$$

$$(c-Y+2)\ \frac{\rho}{c}\ \frac{\rho^{Y}\ (c-Y+1)!\ \rho^{c-Y+2}}{Y!\ c\ (Y+1)_{c-Y}}\ c\,P_0$$

$$= \frac{\rho^{Y}\ (c-Y+2)!\ \rho^{c-Y+2}}{Y!\ c^2\ (Y+1)_{c-Y}}\ P_0\ \left[(c-Y+3)\ \rho + c - c\ \right]$$

$$c\,P_{c+3} = \frac{\rho^{Y}\,(c-Y+2)!\;\rho^{c-Y+2}}{Y!\;c^{2}\;(Y+1)_{c-Y}}\;P_0\;\;(c-Y+3)\,\rho$$

$$c\,P_{c+3} = \frac{\rho^{Y}\,(c-Y+3)!\;\rho^{c-Y+3}}{Y!\;c^{2}\;(Y+1)_{c-Y}}\;P_0$$

$$P_{c+3} = \frac{\rho^{Y}\,(c-Y+3)!\;\rho^{c-Y+3}}{Y!\;c^{3}\;(Y+1)_{c-Y}}\;P_0 = \frac{(c-Y+3)\;(c-Y+2)\;(c-Y+1)\;\lambda^{3}}{(c\mu)^{3}}\,P_c$$

$$= \frac{\rho^{Y}}{Y!}\;\frac{(c-Y+3)\;(c-Y+2)\;(c-Y+1)\;(c-Y)!\;\rho^{c-Y+3}}{c^{3}\;(Y+1)_{c-Y}}\;P_0$$

Similarly,

$$P_n = P_{c+(n-c)} = \frac{\rho^{Y}\,(c-Y+n-c)!\;\rho^{c-Y+n-c}}{Y!\;c^{n-c}\;(Y+1)_{c-Y}}$$

$$P_n = \frac{\rho^{c}\,(n-Y)!}{Y!\;\;(Y+1)_{c-Y}}\left(\frac{\rho}{c}\right)^{n-c}\;P_0$$

$$P_n = \frac{\rho^{Y}}{Y!}\frac{(c-Y+1)_{n-c}\;(c-Y)!\;\;\rho^{c-Y+n-c}}{c^{n-c}\;\;(Y+1)_{c-Y}}\;P_0$$

$$= \frac{\rho^{c}}{Y!}\frac{(c-Y+1)_{n-c}\;(c-Y)!}{(Y+1)_{c-Y}}\;\left(\frac{\rho}{c}\right)^{n-c}$$

$$P_n = \frac{\rho^{Y}}{Y!}\frac{(c-Y)!\;\;\rho^{c-Y}}{(Y+1)_{c-Y}}\;\;\frac{(c-Y+1)_{n-c}\;\lambda^{n-c}}{(c\mu)^{n-c}} \qquad (2.4.13)$$

The empty system probability P_0, can be found from the boundary condition: $\sum_{n=0}^{\infty} P_n = 1$

$$P_0^{-1} = \sum_{n=0}^{Y-1} \frac{\rho^n}{n!} + \frac{\rho^Y}{Y!} \sum_{n=Y}^{c-1} \frac{(n-Y)!\, \rho^{n-Y}}{(Y+1)_{n-Y}} +$$

$$\sum_{n=c}^{\infty} \frac{\rho^c (c-Y+1)_{n-c} (c-Y)!}{Y!\,(Y+1)_{c-Y}} \left(\frac{\rho}{c}\right)^{n-c} \qquad (2.4.14)$$

$$(I)\ \frac{\rho^Y}{Y!} \sum_{n=Y}^{c-1} \frac{(n-Y)!\, \rho^{n-Y}}{(Y+1)_{n-Y}} \qquad n \to n+Y$$

$$\frac{\rho^Y}{Y!} \sum_{n=0}^{c-1-Y} \frac{n!\, \rho^n}{(Y+1)_n}$$

$$\frac{\rho^Y}{Y!} \sum_{n=0}^{c-1-Y} \frac{n!\, \rho^n\, n!}{(Y+1)_n\, n!}$$

$$\frac{\rho^Y}{Y!}\ {}_2F_1(1,1;Y+1;\rho)$$

$$(II)\ \sum_{n=c}^{\infty} \frac{\rho^c (c-Y+1)_{n-c} (c-Y)!}{Y!\,(Y+1)_{c-Y}} \left(\frac{\rho}{c}\right)^{n-c}$$

$$\frac{\rho^c}{Y!} \sum_{n=c}^{\infty} \frac{(c-Y)!}{(Y+1)_{c-Y}} (c-Y+1)_{n-c} \left(\frac{\rho}{c}\right)^{n-c} \qquad n \to n+c$$

$$\frac{\rho^c (c-Y)!}{Y!\,(Y+1)_{c-Y}} \sum_{n=0}^{\infty} \frac{(c-Y+1)_n \left(\frac{\rho}{c}\right)^n n!}{n!}$$

$$\frac{\rho^c (c-Y)!}{Y!\,(Y+1)_{c-Y}} \sum_{n=0}^{\infty} \frac{(1)_n (c-Y+1)_n \left(\frac{\rho}{c}\right)^n}{n!}$$

$$\frac{\rho^c}{Y!} \frac{(c-Y)!}{(Y+1)_{c-Y}}\ {}_2F_0\left(1, c-Y+1; -; \frac{\rho}{c}\right)$$

Hence

$$P_0^{-1} = \sum_{n=0}^{Y-1} \frac{1}{n!} \rho^n + \frac{\rho^Y}{Y!}\ {}_2F_1(1,1;Y+1;\rho) +$$

$$\frac{\rho^c (c-Y)!}{Y!\,(Y+1)_{c-Y}}\ {}_2F_0\left(1, c-Y+1; -; \frac{\rho}{c}\right) \qquad (2.4.15)$$

$$L = \lambda P_0 \frac{\partial}{\partial \lambda} P_0^{-1}$$

$$L = \lambda P_0 \frac{\partial}{\partial \lambda} \left[\sum_{n=0}^{Y-1} \frac{\rho^n}{n!} + \frac{\rho^Y}{Y!} \; {}_2F_1 (1,1;Y+1;\rho) + \right.$$

$$\left. \frac{\rho^c (c-Y)!}{Y! \, (Y+1)_{c-Y}} \; {}_2F_0 \left(1, c-Y+1; -; \frac{\rho}{c}\right) \right]$$

$$(I) \; \lambda P_0 \frac{\partial}{\partial \lambda} \sum_{n=0}^{Y-1} \frac{\rho^n}{n!}$$

$$\lambda P_0 \frac{\partial}{\partial \lambda} \sum_{n=0}^{Y-1} \frac{1}{\mu^n} \frac{\lambda^n}{n!}$$

$$\lambda P_0 \sum_{n=0}^{Y-1} \frac{n \lambda^{n-1}}{\mu^n \, n!}$$

$$P_0 \sum_{n=0}^{Y-1} \frac{\rho^n}{(n-1)!}$$

$$P_0 \sum_{n=1}^{Y-1} \frac{\rho^n}{(n-1)!}$$

$$(II) \lambda P_0 \frac{\partial}{\partial \lambda} \left[\frac{\rho^Y}{Y!} \; {}_2F_1 (1,1;Y+1;\rho) \right] =$$

$$\lambda P_0 \left[{}_2F_1 (1,1;Y+1;\rho) \frac{\partial}{\partial \lambda} \frac{\rho^Y}{Y!} + \right.$$

$$\left. \frac{\rho^Y}{Y!} \frac{\partial}{\partial \lambda} \; {}_2F_1 (1,1;Y+1;\rho) \right]$$

$$(II) \; a \; {}_2F_1 (1,1;Y+1;\rho) \, \lambda P_0 \frac{\partial}{\partial \lambda} \left[\frac{1}{Y!} \frac{\lambda^Y}{\mu^Y} \right] +$$

$$\frac{\rho^Y}{Y!} \lambda P_0 \frac{\partial}{\partial \lambda} \sum_{n=0}^{\infty} \frac{(1)_n \, (1)_n \, (\rho)^n}{(Y+1)_n \, n!}$$

$$ {}_2F_1\,(1,1;Y+1;\rho)\,\lambda\,P_0 \left[\frac{1}{Y!}\;\frac{Y\lambda^{Y-1}}{\mu^{Y}}\right]+ $$

$$ \frac{\rho^{Y}}{Y!}\;\lambda P_0 \sum_{n=0}^{\phi} \frac{(1)_n\,(1)_n\,\frac{1}{\mu^n}\,n\,\lambda^{n-1}}{(Y+1)_n\;n!} $$

$$ {}_2F_1\,(1,1;Y+1;\rho)\frac{\rho^{Y}}{(Y-1)!}\,P_0 + \frac{\rho^{Y}}{Y!}\,P_0 \sum_{n=0}^{\phi}\frac{(1)_n\,(1)_n\,\rho^{n}}{(Y+!)_n\;(n-1)!} $$

$$ \frac{\rho^{Y}}{(Y-1)!}\,{}_2F_1\,(1,1;Y+1;\rho)\,P_0 + \frac{\rho^{Y}\,1.1.\rho}{Y!\,(Y+1)}\,P_0 \sum_{n=1}^{\phi}\frac{(2)_{n-1}\,(2)_{n-1}\,\rho^{n-1}}{(Y+2)_{n-1}\;(n-1)!} $$

$$ \frac{\rho^{Y}}{(Y-1)!}\,{}_2F_1\,(1,1;Y+1;\rho)\,P_0 + \frac{\rho^{Y+1}}{(Y+1)}\,P_0 \sum_{n=0}^{\phi}\frac{(2)_n\,(2)_n\,\rho^{n}}{(Y+2)_n\;n!} $$

$$ \frac{\rho^{Y}}{(Y-1)!}\,{}_2F_1\,(1,1;Y+1;\rho)\,P_0 + \frac{\rho^{Y+1}}{(Y+1)}\,P_0\;{}_2F_1\,(2,2;Y+2;\rho) $$

Hence.

$$ \lambda P_0\,\frac{\partial}{\partial\lambda}\left[\frac{\rho^{Y}}{Y!}\,{}_2F_1\,(1,1;Y+1;\rho)\right]= $$

$$ P_0\left[\frac{\rho^{Y}}{(Y-1)!}\,{}_2F_1\,(1,1;Y+1;\rho)+\frac{\rho^{Y+1}}{(Y+1)!}\,{}_2F_1\,(2,2;Y+2;\rho)\right] $$

$$ (III)\;\lambda P_0\,\frac{\partial}{\partial\lambda}\left[\frac{\rho^{c}\,(c-Y)!}{Y!\,(Y+1)_{c-Y}}\;{}_2F_0\left(1,c+1-Y;-;\frac{\rho}{c}\right)\right] $$

$$ =\;{}_2F_0\left(1,c+1-Y;-;\frac{\rho}{c}\right)\frac{(c-Y)!}{Y!\,(Y+1)_{c-Y}}\,\lambda P_0\,\frac{\partial}{\partial\lambda}\left(\rho^{c}\right)+ $$

$$ \frac{\rho^{c}\,(c-Y)!}{Y!\,(Y+1)_{c-Y}}\,\lambda P_0\,\frac{\partial}{\partial\lambda}\left[{}_2F_0\left(1,c+1-Y;-;\frac{\rho}{c}\right)\right](R) $$

$$\lambda \frac{\partial}{\partial \lambda}\left[\rho^{c}\right] = \lambda \frac{\partial}{\partial \lambda}\left[\frac{1}{\mu^{c}} \lambda^{c}\right]$$

$$= \lambda \left[c \frac{\lambda^{c-1}}{\mu^{c}}\right]$$

$$= c \rho^{c}$$

$$\lambda \frac{\partial}{\partial \lambda}\left[{}_2F_0\left(1, c+1-Y; -; \frac{\rho}{c}\right)\right]$$

$$\lambda \frac{\partial}{\partial \lambda}\left[\sum_{n=0}^{\infty} \frac{(1)_n \, (c+1-Y)_n}{n!} \left(\frac{\rho}{c}\right)^{n}\right]$$

$$\lambda \sum_{n=0}^{\infty} \frac{(1)_n \, (c+1-Y)_n \; n \, \dfrac{\lambda^{n-1}}{\mu^{n} \, c^{n}}}{n!}$$

$$\sum_{n=0}^{\infty} \frac{(1)_n \, (c+1-Y)_n \left(\dfrac{\rho}{c}\right)^{n}}{(n-1)!}$$

$$1.(c+1-Y) \left(\frac{\rho}{c}\right) \sum_{n=1}^{\infty} \frac{(2)_{n-1} \, (c+2-Y)_{n-1} \left(\dfrac{\rho}{c}\right)^{n}}{(n-1)!}$$

$$(c+1-Y) \left(\frac{\rho}{c}\right) \sum_{n=0}^{\infty} \frac{(2)_{n} \, (c+2-Y)_{n} \left(\dfrac{\rho}{c}\right)^{n}}{(n-1)!}$$

$$(c+1-Y) \left(\frac{\rho}{c}\right) {}_2F_0\left(2, c+2-Y; -; \frac{\rho}{c}\right)$$

Put in (R) we get,

$$ {}_2F_0\left(1, c+1-Y;-;\frac{\rho}{c}\right)\frac{(c-Y)!}{Y!\,(Y+1)_{c-Y}}\,c\,\rho^{c}\,P_0 \;+ $$

$$ \frac{\rho^{c}\,(c-Y)!\,(c+1-Y)\left(\frac{\rho}{c}\right)}{Y!\,(Y+1)_{c-Y}}\;{}_2F_0\left(2, c+2-Y;-;\frac{\rho}{c}\right)P_0 $$

Hence.

$$ \lambda P_0\,\frac{\partial}{\partial\lambda}\left[\frac{\rho^{c}\,(c-Y)!}{Y!\,(Y+1)_{c-Y}}\;{}_2F_0\left(1, c+1-Y;-;\frac{\rho}{c}\right)\right] = $$

$$ \frac{(c-Y)!\,c.\,\rho^{c}}{Y!\,(Y+1)_{c-Y}}P_0\;{}_2F_0\left(1, c+1-Y;-;\frac{\rho}{c}\right)+ $$

$$ \frac{\rho^{c+1}\,(c-Y)!\,(c+1-Y)}{c\,Y!\,(Y+1)_{c-Y}}\,P_0\;{}_2F_0\left(2, c+2-Y;-;\frac{\rho}{c}\right) $$

Hence.

$$ L = P_0\left[\sum_{n=1}^{Y-1}\frac{1}{(n-1)!}\rho^{n}+\frac{\rho^{Y}}{(Y-1)!}\;{}_2F_1\,(1,1;Y+1;-;\rho)+\right. $$

$$ \frac{\rho^{Y+1}}{(Y+1)!}\;{}_2F_1\,(2,2;Y+2;-;\rho)+ $$

$$ \frac{(c-Y)!\,c\,\rho^{c}}{Y!\,(Y+1)_{c-Y}}\;{}_2F_0\left(1, c+1-Y;-;\frac{\rho}{c}\right)+ $$

$$ \left.\frac{(c-Y)!(c+1-Y)\rho^{c+1}}{c\,Y!\,(Y+1)_{c-Y}}\;{}_2F_0\left(2, c+2-Y;-;\frac{\rho}{c}\right)\right] \qquad (2.4.16) $$

Let $\alpha = 0,\ \beta = 1$ and $Y = 0$ then we get,

$$\lambda_n = \begin{cases} (n+1)\,\lambda & 0 \le n < c \\ 0 & n \ge 0 \end{cases}$$

and

$$\mu_n = \begin{cases} n\mu & 0 \le n < c \\ c\,\mu & n \ge 0 \end{cases}$$

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad n = 0 \quad (2.4.17)$$

$$P_n'(t) = -\left[(n+1)\,\lambda + n\,\mu\right] P_n(t) + n\,\lambda P_{n-1}(t) + (n+1)\,\mu P_{n+1}(t) \quad 1 \le n < c \quad (2.4.18)$$

$$P_c'(t) = -\left[(c+1)\,\lambda + c\,\mu\right] P_c(t) + c\,\lambda P_{c-1}(t) + c\mu P_{c+1}(t) \quad n = c \quad (2.4.19)$$

$$P_n'(t) = -\left[(n+1)\,\lambda + c\,\mu\right] P_n(t) + n\,\lambda P_{n-1}(t) + c\,\mu P_{n+1}(t) \qquad c+1 \le n < Y+K \quad (2.4.20)$$

There fore the steady - state probability - difference equations are:

$$-\lambda P_0 + \mu P_1 = 0 \qquad n = 0 \qquad (2.4.21)$$

$$-\left[(n+1)\,\lambda + n\,\mu\right] P_n + n\,\lambda P_{n-1} + (n+1)\,\mu P_{n+1} = 0 \quad 1 \le n < c \quad (2.4.22)$$

$$-\left[(c+1)\,\lambda + c\,\mu\right] P_c + c\,\lambda P_{c-1} + c\mu P_{c+1} = 0 \qquad n = c \quad (2.4.23)$$

$$-\left[(n+1)\,\lambda + c\,\mu\right] P_n + n\,\lambda P_{n-1} + c\,\mu P_{c+1} = 0 \quad c+1 \le n < Y+K \quad (2.4.24)$$

From equation (4.2.21)

$$\mu \, P_1 = \lambda \, P_0$$

$$P_1 = \rho \, P_0$$

From Equation (4.2.22)

$$-\left[(n+1)\,\lambda + n\,\mu\right] P_n + n\,\lambda\, P_{n-1} + (n+1)\,\mu\, P_{n+1} = 0 \qquad 1 \le n < c$$

$$(n+1)\,\mu\, P_{n+1} = \left[(n+1)\,\lambda + n\,\mu\right] P_n - n\,\lambda\, P_{n-1}$$

$$(n+1)\, P_{n+1} = \left[(n+1)\rho + n\right] P_n - n\,\rho\, P_{n-1}$$

Put n = 1

$$2\,P_2 = \left[2\,\rho + 1\right] P_1 - \rho\, P_0$$

$$= (2\,\rho + 1)\,\rho\, P_0 - \rho\, P_0$$

$$2\,P_2 = \rho\, P_0 \left[2\,\rho + 1 - 1\right]$$

$$2\,P_2 = 2\rho^2\, P_0$$

$$P_2 = \rho^2\, P_0$$

Put n = 2

$$3\,P_3 = \left[3\,\rho + 2\right] P_2 - 2\,\rho\, P_1$$

$$= (3\,\rho + 2)\,\rho^2\, P_0 - 2\,\rho\,(\rho\, P_0)$$

$$3\,P_3 = 3\,\rho^3\, P_0 + 2\,\rho^2\, P_0 - 2\,\rho^2\, P_0$$

$$P_3 = \rho^3\, P_0$$

Similarly,

$$P_n = \rho^n\, P_0 \qquad 0 < n < c-1 \qquad (4.2.25)$$

Where $\rho = \lambda/\mu$

From Equation (4.2.23)

$$-\left[(c+1)\,\lambda + c\,\mu\right] P_c + c\,\lambda P_{c-1} + c\,\mu P_{c+1} = 0 \qquad n = c$$

$$c\,\mu P_{c+1} = \left[(c+1)\,\lambda + c\,\mu\right] P_c - c\,\lambda P_{c-1}$$

$$c\ P_{c+1} = \left[(c+1)\,\rho + c\,\right] P_c - c\,\rho P_{c-1}$$

$$= \left[(c+1)\,\rho + c\,\right] \rho^{c}\, P_0 - c\,\rho\ .\rho^{c-1}\, P_{c-1}$$

$$= \rho^{c}\, P_0 \left[(c+1)\,\rho + c\ \text{-} c\right]$$

$$= \ (c+1)\,\rho\,\rho^{c}\, P_0$$

$$P_{c+1} = \left(\frac{c+1}{c}\right) \rho\,\rho^{c}\, P_0$$

From Equation (4.2.24)

$$-\left[(n+1)\,\lambda + c\,\mu\right] P_n + n\,\lambda P_{n-1} + c\,\mu P_{n+1} = 0 \quad c+1 \le n < Y+K$$

$$c\,\mu P_{n+1} = \left[(n+1)\,\lambda + c\,\mu\right] P_n - n\,\lambda P_{n-1}$$

$$c\ P_{n+1} = \left[(n+1)\,\rho + c\,\right] P_n - n\,\rho P_{n-1}$$

Put n = c + 1

$$c\ P_{c+2} = \left[(c+2)\,\rho + c\,\right] P_{c+1} - (c+1)\,\rho\, P_c$$

$$= \left[(c+2)\,\rho + c\,\right] \left(\frac{c+1}{c}\right) \rho\rho^{c}\, P_0 - (c+1)\,\rho\ .\rho^{c}\, P_0$$

$$c\ P_{c+2} \quad = \left(\frac{c+1}{c}\right) \rho.\rho^{c}\, P_0 \left[(c+2)\,\rho + c\ \text{-} c\right]$$

$$P_{c+2} \quad = \left(\frac{c+1}{c}\right) \frac{\rho}{c}\ \rho^{c}\, P_0\, (c+2)\,\rho$$

$$P_{c+2} = (c+1)\,(c+2)\ \ \rho^{c} \left(\frac{\rho}{c}\right)^{2} P_0$$

Put n = c + 2

$$c\ P_{c+3} = \left[(c+3)\,\rho + c\,\right] P_{c+2} - \rho(c+2)\ \ P_{c+1}$$

$$= \left[(c+3)\,\rho + c\,\right] P_{c+2} - \ (c+2)\,\rho\ \ P_{c+1}$$

$$= \left[(c+3)\,\rho + c\,\right]\ (c+1)\ (c+2)\ \rho^{c}\left(\frac{\rho}{c}\right)^{2}\ P_0 -$$

$$(c+2)\,\rho\,.\left(\frac{(c+1)}{c}\right)\rho\,.\,\rho^{c}\,P_0$$

$$= \ (c+1)\ (c+2)\ \rho^{c}\left(\frac{\rho}{c}\right)^{2}\ \left[(c+3)\,\rho + c\ - c\right] P_0$$

$$P_{c+3} \ = \ (c+1)\ (c+2)\ (c+3)\ \rho^{c}\left(\frac{\rho}{c}\right)^{3}\ P_0$$

Similarly,

$$P_n = P_{c+(n-c)} \ = \ (c+1)\ (c+2)\ (c+3) - - \ (c - n)\,\rho^{c}\left(\frac{\rho}{c}\right)^{n-c}\ P_0$$

$$P_n = (c+1)_{n-c}\ \rho^{c}\left(\frac{\rho}{c}\right)^{n-c}\ P_0 \qquad\qquad c \ \le n \le Y + K \quad (4.2.26)$$

The boundary condition $\sum_{n=0}^{\infty} P_n = 1$

$$P_0^{-1} = \sum_{n=0}^{c-1}\rho^{n} + \rho^{c}\sum_{n=c}^{\infty}(c+1)_{n-c}\left(\frac{\rho}{c}\right)^{n-c}$$

$$= \sum_{n=0}^{c-1}\rho^{n} + \rho^{c}\sum_{n=0}^{\infty}\frac{(c+1)_n\left(\frac{\rho}{c}\right)^{n}(1)_n}{n!} \qquad\qquad n \to n+c$$

$$P_0^{-1} = \sum_{n=0}^{c-1}\rho^{n} + \rho^{c}\ \ {}_2F_0\left(1, c+1; -; \frac{\rho}{c}\right) \qquad\qquad (4.2.27)$$

$$L \ = \ \lambda\,P_0\ \frac{\partial}{\partial\,\lambda}\ P_0^{-1}$$

$$= \ \lambda\,P_0\ \frac{\partial}{\partial\,\lambda}\left[\sum_{n=0}^{c-1}\rho^{n} + \rho^{c}\ \ {}_2F_0\left(1, c+1; -; \frac{\rho}{c}\right)\ \right]$$

$$= P_0\ \left[\sum_{n=0}^{c-1}\frac{n\,\lambda^{\,n-1}}{\mu^{\,n}}\ \lambda + \ {}_2F_0\left(1, c+1; -; \frac{\rho}{c}\right)\lambda\,\frac{1}{\mu^{c}}\ \ c.\ \lambda^{\,c-1} + \right.$$

$$+\lambda\rho^{c}\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{\infty}\frac{(1)_n\,(c+1)_n\left(\frac{\rho}{c}\right)^{n}}{n!}\right]$$

$$= P_0\left[\sum_{n=0}^{c-1} n.\,\rho^{n} + {}_2F_0\left(1,c+1;-;\frac{\rho}{c}\right)c.\,\rho^{c} + \right.$$

$$\left.\rho^{c}\sum_{n=0}^{\infty}\frac{(1)_n\,(c+1)_n.\,n\left(\frac{\rho}{c}\right)^{n}}{n!}\right]$$

$$= P_0\left[\sum_{n=0}^{c-1} n.\,\rho^{n} + {}_2F_0\left(1,c+1;-;\frac{\rho}{c}\right)c.\,\rho^{c} + \right.$$

$$\rho^{c}\sum_{n=0}^{\infty}\frac{(1)_n\,(c+1)_n\left(\frac{\rho}{c}\right)^{n}}{(n-1)!}$$

$$L = P_0\left[\sum_{n=0}^{c-1} n.\,\rho^{n} + {}_2F_0\left(1;c+1;-;\frac{\rho}{c}\right)c.\,\rho^{c} + \right.$$

$$\rho^{c}.1.\,(c+1)\left(\frac{\rho}{c}\right)\sum_{n=1}^{\infty}\frac{(2)_{n-1}\,(c+2)_{n-1}\left(\frac{\rho}{c}\right)^{n-1}}{(n-1)!}$$

$$L = P_0\left[\sum_{n=0}^{c-1} n.\,\rho^{n} + c\,\rho^{c}\;{}_2F_0\left(1;c+1;-;\frac{\rho}{c}\right)c.\,\rho^{c} + \right.$$

$$(c+1)\,\frac{\rho^{c+1}}{c}.\;{}_2F_0\left(2;c+2;-;\frac{\rho}{c}\right) \qquad (4.2.28)$$

The machine availability (rate of production per Machine) is.

$$\text{M.A.} = 1 - L/K$$

The operative efficiency (utilization) is.

$$\text{O. E.} = 1 - \sum_{n=0}^{C-1}(1-n/c)\,P_n$$

Chapter - 3

The Transient Behaviour of the Machine Interference M/M/C/K/N (with balking, reneging and spares)

INTRODUCTION :

In this chapter we consider a queueing model M/M/C/K/N with balking, reneging and spares and we assume a finite source (population) of N customers, c servers are available, customer's arriving rate is λ. We take Y spares so that when a machine fails, a spares is immediately substituted for it, if it happens that all spares are used and a breakdown occurs then the system becomes short. When a machine is repaired it then becomes a spares (unless the system is short in which case the repaired machine goes immediately into service). We analyze the two cases of spares, first of them is $C \leq Y$ and the other $C > Y$.

We assume that a queueing model M/M/C/K/N for transient behaviour with balking, reneging and spares. The arrival and service rates are λ_n and μ_n respectively. The important works on machine repair problem have been considered by many researchers. Kleinrock [98] derived M/M/C/K/N for machine interference but without assumption of balking, reneging or spares. Kness have mentioned the transient behaviour of the repairman problem by using singular perturbation method to scale equations for the number of failed machine. Mokaddis *et. al.* discussed the cost analysis for two units warm stand by system with two types of repair facilities. Jain and Singh studied finite queueing model with random failure and delayed repairs.

In many realistic manufacturing / production situation, due to long backlog of failed units. The units may be discouraged i.e. either balked or reneged. Highighi et. al.[78] analyzed a multiserver Markovian queueing model with balking and reneging. The M/M/1/N queue with general balk function, reneging and an additional server for longer queue was studied by Abou-El-Ata and Kote [2]. Abou – El – Ata and Hariri [4] considered more general model M/M/C/N queue with balking and reneging. Abou – El – Ata and Showky [3] discussed the analytical solution of the single server Markovian over flow queue with balking, reneging and additional server for longer queues and A.I Showky [132] considered M/M/C/K/N machine interference model with balking, reneging and spares.

We develop transient behaviour of the machine interference : M/M/C/K/N(with balking, reneging and spares) by using birth death process and Laplace transform technique and used boundary condition. We discuss two cases $Y \geq C$and $Y \leq C$ and also discuss the special case by using the hypergeometric distribution of first kind and second kind.

3.1 CASE – I

In this case, we study the transient solution of machine interference system : M/M/C/K/N with balking reneging and spares Y $\geq$ C. then the set of birth death coefficients are as follows :-

We consider :

$$\lambda_n = \begin{cases} N\lambda & 0 \leq n < C \\ N\beta\lambda & C \leq n < Y \\ (N+Y-n)\beta\lambda, & Y \leq n < Y+K \\ 0, & n \geq Y+K \end{cases}$$

and

$$\mu_n = \begin{cases} n\mu, & 0 \leq n \leq C \\ C\mu + (n-C)\alpha, & C+1 \leq n < Y+K \end{cases}$$

In the usual arguments of the δ - technique the probability differential difference equations are :

$$P_0'(t) = -N\lambda P_0(t) + \mu P_1(t), \quad n = 0 \tag{3.1.1}$$

$$P_n'(t) = -(N\lambda + n\mu) P_n(t) + N\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t), \quad 1 \leq n < C \tag{3.1.2}$$

$$P_c'(t) = -(N\beta\lambda + C\mu) P_c(t) + N\lambda P_{c-1}(t) + (C\mu + \alpha) P_{c+1}(t), \quad n = C \tag{3.1.3}$$

$$P_n'(t) = -(N\beta\lambda + C\mu + (n-C)\alpha) P_n(t) + N\beta\lambda P_{n-1}(t) +$$

$$[C\mu + (n+1-C)\alpha] P_{n+1}(t), \quad C+1 \leq n < Y \tag{3.1.4}$$

$$P_n'(t) = -[(N+Y-n)\beta\lambda + c\mu + (n-c)\alpha] P_n(t) + (N+Y-n+1)\beta\lambda P_{n-1}(t) +$$

$[C\mu+(n+1-C)\alpha]\, P_{n+1}(t),\ \ Y+1 \le n < Y+K$ (3.1.5)

$P'_{Y+K}(t) = -[C\mu + (Y+K-C)\,\alpha\,]\, P_{Y+K}(t) + (N-K+1)\,\beta\lambda\, P_{Y+K-1}(t),\ \ n = Y+K$

(3.1.6)

Taking the laplace transform of the differential – difference equations then we have.

$$S\,\bar{P}_0(s) - P(o) = -N\lambda\,\bar{P}_0(s) + \mu\,\bar{P}_1(s),\quad n = 0 \tag{3.1.7}$$

$$S\,\bar{P}_n(s) - P_n(o) = -(N\lambda + n\mu)\,\bar{P}_n(s) + N\lambda\,\bar{P}_{n-1}(s) + (n+1)\,\mu\,\bar{P}_{n+1}(s)$$

$$1 \le n < C \quad (3.1.8)$$

$$S\,\bar{P}_c(s) - P_c(o) = -(N\beta\lambda + C\mu)\,\bar{P}_c(s) + N\lambda\,\bar{P}_{c-1}(s) + (C\mu + \alpha)\,\bar{P}_{C+1}(s),\ n = C$$

(3.1.9)

$$S\,\bar{P}_n(s) - P_n(o) = -[N\beta\lambda + C\mu + (n-C)\,\alpha]\,\bar{P}_n(s) + N\beta\lambda\,\bar{P}_{n-1}(s) +$$

$$[C\mu + (n+1-C)\,\alpha]\,\bar{P}_{n+1}(s),\quad C+1 \le n < Y \quad (3.1.10)$$

$$S\,\bar{P}_n(s) - P_n(o) = -[N + Y - n)\,\beta\lambda + C\mu + (n-C)\,\alpha]\,\bar{P}_n(s) + (N+Y-n+1)$$

$$\beta\lambda\,\bar{P}_{n-1}(s) + [C\mu + (n+1-C)\,\alpha]\,\bar{P}_{n+1}(s), Y+1 \le n<Y+K \quad (3.1.11)$$

$$S\,\bar{P}_{Y+K}(s) - P_{Y+K}(o) = -[C\mu + (Y+K-C)\,\alpha]\,\bar{P}_{Y+K}(s) + (N-K+1)\,\beta\lambda\,\bar{P}_{Y+K-1}(s)$$

$$N=Y+K \quad (3.1.12)$$

Define boundary condition

$$P_n(o) = S\,\bar{P}_{n-1}(s) \tag{3.1.13}$$

From equation (3.1.7)

$$S\,\bar{P}_0(s) - P(o) = -N\lambda\,\bar{P}_0(s) + \mu\,\bar{P}_1(s)$$

$\mu\ \bar{P}_1(s) = (N\lambda + S)\ \bar{P}_0(s)$

$\bar{P}_1(s) = (N\lambda/\mu + S/\mu)\ \bar{P}_0(s)$

$\bar{P}_1(s) = (N\rho + \phi)\ \bar{P}_0(s)$ $\qquad\qquad \rho = \lambda/\mu,\ \phi = S/\mu$

From equation (3.1.8)

$S\ \bar{P}_n(s) - P_n(o) = -(N\lambda + n\mu)\ \bar{P}_n(s) + N\lambda\ \bar{P}_{n-1}(s) + (n+1)\ \mu\ \bar{P}_{n+1}(s)$

$(n+1)\mu\ \bar{P}_{n+1}(s) = (N\lambda + n\mu + S)\ \bar{P}_n(s) - N\lambda\ \bar{P}_{n-1}(s) - P_n(o)$

$(n+1)\mu\ \bar{P}_{n+1}(s) = (N\lambda + n\mu + S)\ \bar{P}_n(s) - N\lambda\ \bar{P}_{n-1}(s) - S\ \bar{P}_{n-1}(s)$

$(n+1)\mu\ \bar{P}_{n+1}(s) = (N\lambda + n\mu + S)\ \bar{P}_n(s) - (N\lambda + S)\ \bar{P}_{n-1}(s)$

$(n+1)\ \bar{P}_{n+1}(s) = (N\rho + n + \phi)\ \bar{P}_n(s) - (N\rho + \phi)\ \bar{P}_{n-1}(s)$

Put, $n = 1$

$2\ \bar{P}_2(s) = (N\rho + 1 + \phi)\ \bar{P}_1(s) - (N\rho + \phi)\ \bar{P}_0(s)$

$2\ \bar{P}_2(s) = (N\rho + \phi + 1)(N\rho + \phi)\ \bar{P}_0(s) - (N\rho + \phi)\ \bar{P}_0(s)$

$= (N\rho + \phi)(N\rho + \phi + 1 - 1)\ \bar{P}_0(s)$

$$\bar{P}_2(s) = \frac{(N\rho+\phi)^2}{2!}\bar{P}_0(s)$$

Put, $n = 2$

$3\ \bar{P}_3(s) = (N\rho + 2 + \phi)\ \bar{P}_2(s) - (N\rho + \phi)\ \bar{P}_1(s)$

$$3\ \bar{P}_3(s) = (N\rho + \phi + 2)\frac{(N\rho+\phi)^2}{2!}\bar{P}_0(s) - (N\rho + \phi) . (N\rho + \phi)\bar{P}_0(s)$$

$$= \frac{(N\rho+\phi)^2}{2!}\bar{P}_0(s)\ (N\rho + \phi + 2 - 2\ )$$

$$3\ \overline{P}_3(s) = \frac{(N\rho+\phi)^3}{2!}\overline{P}_0(s)$$

$$\overline{P}_3(s) = \frac{(N\rho+\phi)^3}{3!}\overline{P}_0(s)$$

Similarly

$$\overline{P}_n(s) = \frac{(N\rho+\phi)^n}{n!}\overline{P}_0(s),\ 0\le n\le c \tag{3.1.14}$$

Where, $\rho = \lambda/\mu$, $\phi = S/\mu$

From equation (3.1.9)

$$S\ \overline{P}_c(s) - P_c(o) = -(N\beta\lambda + C\mu)\ \overline{P}_c(s) + N\lambda\ \overline{P}_{c-1}(s) + (C\mu+\alpha)\ \overline{P}_{C+1}(s)$$

$$(C\mu+\alpha)\ \overline{P}_{c+1}(s) = (N\beta\lambda + C\mu + S)\ \overline{P}_c(s) - N\lambda\ \overline{P}_{c-1}(s) - P_c(o)$$

$$= (N\beta\lambda + C\mu + S)\ \overline{P}_c(s) - (N\lambda + S)\ \overline{P}_{c-1}(s)$$

$$(\delta+1)\ \overline{P}_{c+1}(s) = (N\beta\gamma + \delta + \psi)\ \overline{P}_c(s) - (N\gamma + \psi)\ \overline{P}_{c-1}(s)$$

Where, $\lambda/\alpha = \gamma$, $C\mu/\alpha = \delta$, $s/\alpha = \psi$. $S/\mu = \phi$

$$\overline{P}_{c+1}(s) = \frac{(N\beta\gamma+\delta+\psi)}{\delta+1}\overline{P}_c(s)\ \frac{(N\gamma+\psi)}{\delta+1}\ \overline{P}_{c-1}(s)$$

$$\overline{P}_{c+1}(s) = \frac{(N\beta\gamma+\delta+\psi)}{\delta+1}\ \frac{(N\rho+\phi)^c}{C!}\ \overline{P}_0(s) - \frac{(N\gamma+\psi)}{\delta+1}\ \frac{(N\rho+\phi)^{c-1}}{(C-1)!}\ \overline{P}_0(s)$$

$$= \frac{(N\beta\gamma+\delta+\psi)}{\delta+1}\ \frac{(N\rho+\phi)^c}{C!}\ \overline{P}_0(s) - \frac{(N\lambda/\alpha+S/\alpha)}{\delta+1}\ \frac{(N\rho+\phi)^{c-1}}{(C-1)!C}C\ \overline{P}_0(s)$$

$$= \frac{(N\beta\gamma+\delta+\psi)}{\delta+1}\ \frac{(N\rho+\phi)^c}{C!}\ \overline{P}_0(s) - \frac{(N\lambda+S)}{\alpha(\delta+1)}\ \frac{(N\rho+\phi)^{c-1}}{C!}C\ \overline{P}_0(s)$$

$$= \left(\frac{N\beta\gamma+\delta+\psi}{\delta+1}\right)\frac{(N\rho+\phi)^c}{C!}\,\overline{P}_0(s) - \frac{(N\rho+\phi)}{\delta+1}\,\frac{(N\rho+\phi)^{c-1}}{C!}\,\frac{c\mu}{\alpha}\,\overline{P}_0(s)$$

$$= \left(\frac{N\beta\gamma+\delta+\psi}{\delta+1}\right)\frac{(N\rho+\phi)^c}{C!}\,\overline{P}_0(s) - \frac{(N\rho+\phi)^c}{C!}\,\frac{\delta}{\delta+1}\,\overline{P}_0(s)$$

$$= \frac{(N\rho+\phi)^c}{C!}\,\frac{1}{\delta+1}\,\overline{P}_0(s)\,[N\beta\gamma+\psi+\delta-\delta]$$

$$\overline{P}_{c+1}(s) = \frac{(N\rho+\phi)^c}{C!}\,\overline{P}_0(s)\left(\frac{N\beta\gamma+\psi}{\delta+1}\right)$$

From equation (3.1.10)

$$S\,\overline{P}_n(s) - P_n(o) = -[N\beta\lambda + C\mu + (n-C)\alpha]\,\overline{P}_n(s) + N\beta\lambda\,\overline{P}_{n-1}(s) +$$

$$[C\mu + (n+1-C)\,\alpha]\,\overline{P}_{n+1}(s) \qquad [C\mu + (n+1-C)\,\alpha]\,\overline{P}_{n+1}(s) = [N\beta\lambda + C\mu + (n-C)\alpha+S]$$

$$\overline{P}_n(s) - N\beta\lambda\,\overline{P}_{n-1}(s) - P_n(o)$$

$$[C\mu + (n+1-C)\,\alpha]\,\overline{P}_{n+1}(s) = [N\beta\lambda + C\mu + (n-C)\alpha+S]\,\overline{P}_n(s) - (N\beta\lambda+S)\,\overline{P}_{n-1}(s)$$

$$[\delta + n+1 - C]\,\overline{P}_{n+1}(s) = [N\beta\gamma + \delta + n - C + \Psi]\,\overline{P}_n(s) - [N\beta\gamma + \Psi]\,\overline{P}_{n-1}(s)$$

Put n=c+1

$$[\delta + c + 1 + 1 - c]\overline{P}_{c+2}(s) = [N\beta\gamma + \delta + c + 1 - c + \psi]\overline{P}_{c+}(s) - (N\beta\gamma + \psi)\overline{P}_c(s)$$

$$[\delta + 2]\,\overline{P}_{c+2}(s) = [N\,\beta+\delta + \Psi + 1]\,\overline{P}_{c+1}(s) - [N\beta\gamma + \Psi]\,\overline{P}_c(s)$$

$$[N\beta\gamma + \delta + \Psi + 1]\,\frac{(N\rho+\phi)^c}{C!}\left(\frac{N\beta\gamma+\psi}{\delta+1}\right)\overline{P}_0(s) - [N\beta\gamma+\psi]\frac{(N\rho+\phi)^c}{C!}\,\overline{P}_0(s)$$

$$= [N\beta\gamma+\psi]\,\frac{(N\rho+\phi)^c}{C!}\,\overline{P}_0(s)\left[\frac{N\beta\gamma+\delta+\psi+1}{\delta+1} - 1\right]$$

$$= \frac{(N\rho+\phi)^c}{C!}\,\overline{P}_0(s)\,[N\beta\gamma+\psi]\left[\frac{N\beta\gamma+\delta+\psi+1-\delta-1}{\delta+1}\right]$$

$$= \frac{(N\rho+\phi)^c}{C!}\frac{(N\beta\gamma+\psi)^2}{\delta+1}\overline{P}_0(s)$$

$$\overline{P}_{c+2}(s) = \frac{(N\rho+\phi)^c}{C!}\frac{(N\beta\gamma+\psi)^2}{(\delta+1)(\delta+2)}\overline{P}_0(s)$$

Put n = C + 2

$$[\delta + C+2 +1-C] \ \overline{P}_{c+3}(s) = (N\beta\gamma +\delta+C+2- C+\Psi) \ \overline{P}_{c+2}(s) - (N\beta\gamma +\Psi) \ \overline{P}_{c+1}(s)$$

$$Y + 1 \leq n < Y + K$$

$$[\delta + 3] \ \overline{P}_{c+3}(s) = (N\beta\gamma +\delta+2+\Psi) \ \overline{P}_{c+2}(s) - (N\beta\gamma +\Psi) \ \overline{P}_{c+1}(s)$$

$$= (N\beta\gamma +\delta+2+\Psi) \frac{(N\rho+\phi)^c}{C!} \frac{(N\beta\gamma+\psi)^2}{(\delta+1)(\delta+2)}\overline{P}_0(s) -$$

$$(N\beta\gamma + \Psi) \quad \frac{(N\rho+\phi)^c}{C!}\frac{(N\beta\gamma+\psi)}{(\delta+1)}\overline{P}_0(s)$$

$$= \frac{(N\rho+\phi)^c}{C!}\frac{(N\beta\gamma+\psi)^2}{(\delta+1)(\delta+2)}\overline{P}_0(s) \ (N\beta\gamma +\delta+2+\Psi-\delta-2)$$

$$= \frac{(N\rho+\phi)^c}{C!}\frac{(N\beta\gamma+\psi)^3}{(\delta+1)(\delta+2)}$$

$$\overline{P}_{c+3}(s) = \frac{(N\rho+\phi)^c}{C!}\frac{(N\beta\gamma+\psi)^3}{(\delta+1)(\delta+2)(\delta+3)}\overline{P}_0(s)$$

Similarly

$$\overline{P}_n(s) = \overline{P}_{c+(n-c)}(s) = \frac{(N\rho+\phi)^c}{C!}\frac{(N\beta\gamma+\psi)^{n-c}}{(\delta+1)_{n-c}}\overline{P}_0(s), \ C + 1 \leq n < Y \quad (3.1.15)$$

From equation (3.1.11)

$$S \ \overline{P}_n(s) - P_n(o) = -[(N+ Y-n) \beta\lambda + C\mu + (n-C)\alpha] \ \overline{P}_n(s) + (N+Y-n+1) \beta\lambda \ \overline{P}_{n-1}(s)$$

$$+ [C\mu + (n+1-C)\alpha] \ \bar{P}_{n+1}(s)$$

$$[C\mu + (n+1-C)\alpha] \ \bar{P}_{n+1}(s) = [(N + Y-n) \beta\lambda + C\mu + (n-C)\alpha + S] \ \bar{P}_n(s) -$$

$$(N + Y - n + 1) \beta\lambda \ \bar{P}_{n-1}(s) - P_n(o)$$

$$(\delta+n+1-C) \ \bar{P}_{n+1}(s) = [(N+Y-n) \beta\gamma + \delta + n-C+\Psi] \ \bar{P}_n(s) - [(N+Y-n+1) \beta\gamma + \Psi] \ \bar{P}_{n-1}(s)$$

Put $n = Y + 1$

$$(\delta+Y+1+1-C) \ \bar{P}_{Y+1+1}(s) = [(N+Y-Y-1) \beta\gamma + \delta + Y+1-C+\Psi] \ \bar{P}_{Y+1}(s) -$$

$$[(N+Y-Y-1+1) \beta\gamma + \Psi] \ \bar{P}_{Y+1-1}(s)$$

$$(\delta+Y+2-C) \ \bar{P}_{Y+2}(s) = [(N-1) \beta\gamma + \delta + Y+1-C+\Psi] \ \bar{P}_{Y+1}(s) - [(N\beta\gamma + \Psi] \ \bar{P}_Y(s)$$

$$= [(N-1) \beta\gamma + \delta + Y+1-C+\Psi] \ \frac{(N\rho+\phi)^C}{C!} \frac{(N\beta\gamma+\psi)^{Y+1-C}}{(\delta+1)_{Y+1-C}} \ \bar{P}_0(s)$$

$$- \ (N\beta\gamma + \Psi) \ \frac{(N\rho+\phi)^C}{C!} \frac{(N\beta\gamma+\psi)^{Y-C}}{(\delta+1)_{Y-C}} \ \bar{P}_0(s)$$

$$= \frac{(N\rho+\phi)^C}{C!} \frac{(N\beta\gamma+\psi)^{Y+1-C}}{(\delta+1)_{Y-C}} \ \bar{P}_0(s) \left[\frac{(N-1)\beta\gamma + \delta + Y + 1 - C + \psi}{\delta + Y + 1 - C} - 1 \right]$$

$$= \frac{(N\rho+\phi)^C}{C!} \frac{(N\beta\gamma+\psi)^{Y+1-C}}{(\delta+1)_{Y-C}} \ \bar{P}_0(s)$$

$$\left[\frac{(N-1)\beta\gamma+\delta+Y+1-C+\psi-\delta-Y-1+C}{\delta+Y+1-C} \right]$$

$$= \frac{(N\rho+\phi)^C}{C!} \frac{(N\beta\gamma+\psi)^{Y+C-3}}{(\delta+1)_{Y-C}} \ \bar{P}_0(s) \left[\frac{(N-1)\beta\gamma+\psi}{\delta+Y+1-C} \right]$$

$$\bar{P}_{Y+2}(s) = \frac{(N\rho+\phi)^C}{C!} \frac{(N\beta\gamma+\psi)^{Y-C}}{(\delta+1)_{Y-C}} \frac{(N\beta\gamma+\psi)[(N-1)\beta\gamma+\psi]}{(\delta+Y+1-C)(\delta+Y+2-C)} \ \bar{P}_0(s)$$

Put n = Y+2

$(\delta+Y+2+1-C)\ \bar{P}_{y+3}(s) = [\ (N + Y- Y-2)\ \beta\gamma + \delta + Y+ 2 - C + \psi]\ \ \bar{P}_{y+2}(s) -$

$[(N + Y-Y- 2 + 1)\ \beta\ \gamma + \psi]\ \ \bar{P}_{y+1}(s)$

$(\delta+Y+3-C)\ \ \bar{P}_{y+3}(s) = [\ (N-2)\ \beta\gamma + \delta + Y+ 2 - C + \psi]\ \ \bar{P}_{y+2}(s) - [(N - 1)\ \beta\gamma\ + \psi]$

$P_{y+1}(s)$

$= [\ (N-2)\ \beta\gamma\ + \delta + Y+ 2 - C + \psi]$

$$\frac{(N\rho+\phi)^C}{C!}\frac{(N\beta\gamma+\psi)^{Y-C}}{(\delta+1)_{Y-C}}\ \frac{(N\beta\gamma+\psi)[(N-1)\beta\gamma+\psi]}{(\delta+Y+1-C)(\delta+Y+2-C)}\quad \bar{P}_0(s) - [(N\text{-}1)\ \beta\gamma + \psi]$$

$$\frac{(N\rho+\phi)^C}{C!}\frac{(N\beta\gamma+\psi)^{Y+1-C}}{(\delta+1)_{Y+1-C}}\quad \bar{P}_0(s)$$

$$[(N\text{-}1)\ \beta\gamma + \psi]\frac{(N\rho+\phi)^C}{C!}\frac{(N\beta\gamma+\psi)^{Y+1-C}}{(\delta+1)_{Y+1-C}}\quad \bar{P}_0(s)\left[\frac{(N-2)\beta\gamma+\psi+\delta+y+2-c}{\delta+y+2-c}-1\right]$$

$$= [(N\text{-}1)\ \beta\ \gamma + \psi]\frac{(N\rho+\phi)^C}{C!}\frac{(N\beta\gamma+\psi)^{Y+1-C}}{(\delta+1)_{Y-C+1}}\qquad \frac{[(N-2)\beta\gamma+\psi]}{\delta+Y+2-C}\ \bar{P}_0(s)$$

$$\bar{P}_{y+3}(s) = \frac{(N\rho+\phi)^C}{C!}\frac{(N\beta\gamma+\psi)^{Y-C}}{(\delta+1)_{Y-c}}\ \frac{(N\beta\gamma+\psi)[(N-1)\beta\gamma+\psi][(N-2)\beta\gamma+\psi]}{(\delta+Y+1-C)(\delta+Y+2-C)(\delta+Y+3-C)}\quad \bar{P}_0(s)$$

$$\bar{P}_{Y+3} = \frac{(N\rho+\phi)^C(N\beta\gamma+\psi)^{Y-C}}{C!(\delta+1)_{Y-C}}\ \frac{(\beta\gamma)^3[N+\frac{\psi}{\beta\gamma}][(N-1)+\frac{\psi}{\beta\gamma}][(N-2)+\frac{\psi}{\beta\gamma}]}{(\delta+Y+1-C)(\delta+Y+2-C)(\delta+Y+3-C)}\ \bar{P}_0(s)$$

Similarly

$$\bar{P}_n(s) = \ \bar{P}_{Y+(n-Y)}(s) = \frac{(N\rho+\phi)^C(N\beta\gamma+\psi)^{Y-C}}{C!(\delta+1)_{Y-C}}\ \frac{(\beta\gamma)^{n-Y}[N+\frac{\psi}{\beta\gamma}]_{n-Y}}{(\delta+Y+1-C)_{n-Y}}\ \bar{P}_0(s)$$

$$\overline{P}_n(s) = \frac{(N\rho+\phi)^C (N\beta\gamma+\psi)^{Y-C} (\beta\gamma)^{n-Y} [N+\frac{\psi}{\beta\gamma}]_{n-Y}}{C!(\delta+1)_{Y-C}(\delta+Y+1-c)_{n-Y}} \overline{P}_0(s) \qquad (3.1.16)$$

Where, $[N+\frac{\psi}{\beta\gamma}]_n = [N+\frac{\psi}{\beta\gamma}]\,[(N-1)+\frac{\psi}{\beta\gamma}]\,[(N-2)+\frac{\psi}{\beta\gamma}]\ldots\ldots\ldots\,[(N-n+1)+\frac{\psi}{\beta\gamma}]$

Thus $\overline{P}_n(s)$ can be written as follows :

$$\overline{P}_n(s) = \begin{cases} \dfrac{(N\rho+\phi)^n}{n!}\overline{P}_0(s), & 0 \le n < c-1 \\[2ex] \dfrac{(N\rho+\phi)^c}{C!}\dfrac{(N\beta\gamma+\psi)^{n-c}}{(\delta+1)_{n-c}}\overline{P}_0(s), & c \le n < Y \\[2ex] \dfrac{(N\rho+\phi)^c (N\beta\gamma+\psi)^{Y-c} (\beta\gamma)^{n-Y}\left[N+\dfrac{\psi}{\beta\gamma}\right]_{n-Y}}{C!(\delta+1)_{Y-c}(\delta+Y+1-c)_{n-Y}}\overline{P}_0(s), & Y+1 \le n < Y+K \end{cases} \qquad \textbf{(3.1.17)}$$

To find P_0, the boundary condition: $\sum_{n=0}^{y+k}\overline{P}_n(s) = 1$

$$\sum_{n=0}^{C-1}\frac{(N\rho+\phi)^n}{n!}\overline{P}_0(s) + \sum_{n=c}^{Y}\frac{(N\rho+\phi)^C}{C!}\frac{(N\beta\gamma+\psi)^{n-C}}{(\delta+1)_{n-c}}\overline{P}_0(s) +$$

$$\frac{(N\rho+\phi)^C (N\beta\gamma+\psi)^{Y-C}}{C!(\delta+1)_{Y-C}}\sum_{n=Y+1}^{Y+K}\frac{(\beta\gamma)^{n-Y}\left[N+\dfrac{\psi}{\beta\gamma}\right]_{n-Y}}{(\delta+Y+1-c)_{n-Y}}\overline{P}_0(s) = 1$$

$$\overline{P}_0^{-1}(s) = \sum_{n=0}^{C-1}\frac{(N\rho+\phi)^n}{n!} + \frac{(N\rho+\phi)^C}{C!}\sum_{n=C}^{Y}\frac{(N\beta\gamma+\psi)^{n-C}}{(\delta+1)_{n-C}} +$$

$$\frac{(N\rho+\phi)^C (N\beta\gamma+\psi)^{Y-C}}{C!(\delta+1)_{Y-C}}\sum_{n=Y+1}^{Y+K}\frac{(\beta\gamma)^{n-Y}\left[N+\dfrac{\psi}{\beta\gamma}\right]_{n-Y}}{(\delta+Y+1-c)_{n-Y}} \qquad \textbf{(3.1.18)}$$

$$(\text{I})\ \sum_{n=c}^{Y-C}\frac{(N\beta\gamma+\psi)^{n-C}}{(\delta+1)_{n-C}} \qquad n \longrightarrow n+C$$

$$\sum_{n=0}^{Y-C}\frac{(N\beta\gamma+\psi)^{n}\,n!}{(\delta+1)_{n}\,n!}=\sum_{n=0}^{Y-C}\frac{(N\beta\gamma+\psi)^{n}(1)n}{(\delta+1)_{n}\,n!}= {}_1F_1(1;\delta+1;N\beta\gamma+\Psi)$$

$$(\text{II})\quad \sum_{n=Y+1}^{Y+K}\frac{(N\rho+\phi)^{C}(N\beta\gamma+\psi)^{Y-C}(\beta\gamma)^{n-Y}[N+\frac{\psi}{\beta\gamma}]_{n-Y}}{C!(\delta+1)_{Y-C}(\delta+Y+1-C)_{n-Y}}$$

$$\frac{(N\rho+\phi)^{C}(N\beta\gamma+\psi)^{Y-C}}{C!(\delta+1)_{Y-C}}\sum_{n=Y+1}^{Y+K}\frac{(\beta\gamma)^{n-Y}[N+\frac{\psi}{\beta\gamma}]_{n-Y}}{(\delta+Y+1-C)_{n-Y}}$$

$$\frac{(N\rho+\phi)^{C}(N\beta\gamma+\psi)^{Y-C}}{C!(\delta+1)_{Y-C}}\sum_{n=0}^{K-1}\frac{(\beta\gamma)^{n+1}[N+\frac{\psi}{\beta\gamma}]_{n+1}}{(\delta+Y+1-C)_{n+1}}$$

$$\frac{(N\rho+\phi)^{C}(N\beta\gamma+\psi)^{Y-C}}{C!(\delta+1)_{Y-C}}\sum_{n=0}^{K-1}\frac{[N+\frac{\psi}{\beta\gamma}]_{n+1}(\beta\gamma)^{n+1}}{(\delta+Y+1-C)_{n+1}}$$

$$\frac{(N\rho+\phi)^{C}(N\beta\gamma+\psi)^{Y-C}}{C!(\delta+1)_{Y-C}}\frac{[N+\frac{\psi}{\beta\gamma}](\beta\gamma)}{(\delta+Y+1-C)}\sum_{n=0}^{K-1}\frac{[(N-1)+\frac{\psi}{\beta\gamma}]_{n}(\beta\gamma)^{n}(1)_{n}}{(\delta+Y+2-C)_{n}\,n!}$$

$$\frac{(N\rho+\phi)^{C}(N\beta\gamma+\psi)^{Y-C+1}}{C!(\delta+1)_{Y-C+1}}\,{}_2F_1(1;(N-1)+\frac{\psi}{\beta\gamma};\delta+Y+2-C;\beta\gamma)$$

$$\bar{P}_0^{-1}(s)=\sum_{n=0}^{C-1}\frac{(N\rho+\phi)^{n}}{n!}+\frac{(N\rho+\phi)^{C}}{C!}\sum_{n=C}^{Y}\frac{(N\beta\gamma+\psi)^{n-C}}{(\delta+1)_{n-C}}+$$

$$\frac{(N\rho+\phi)^{C}(N\beta\gamma+\psi)^{Y-C}}{C!(\delta+1)_{Y-C}}\sum_{n=Y+1}^{Y+K}\frac{(\beta\gamma)^{n-Y}[N+\frac{\psi}{\beta\gamma}]_{n-Y}}{(\delta+Y+1-C)_{n-Y}}$$

Hence

$$\overline{P}_0^{-1}(s)=\sum_{n=0}^{C-1}\frac{(N\rho+\phi)^n}{n!}+\frac{(N\rho+\phi)^C}{C!}\,{}_1F_1(1;\delta+1;N\beta\gamma+\Psi)+$$

$$\frac{(N\rho+\phi)^C(N\beta\gamma+\psi)^{Y-C+1}}{C!(\delta+1)_{Y-C+1}}\,{}_2F_1(1;(N-1)+\frac{\psi}{\beta\gamma};\delta+Y+2-C;\beta\gamma)\quad(3.1.19)$$

Where $(1)_n = 0$ when $n>Y-c$ in the first hypergeometric function and either $(1)_n = 0$ or $(1-N)_n=0$ when $n>K-1$ in the second hypergeometric function.

To calculated the expected number of units in the system, a result due to Abou-El-Ata is used. Which states that for a simple birth death process.

$$L=\frac{-\lambda lnP_0}{\partial\lambda}=-\rho\frac{dlnP_0}{d\rho}$$

Thus $L=\lambda\overline{P}_0(s)\dfrac{\partial\overline{P}_0^{-1}(s)}{\partial\lambda}$

$$=\lambda\,\overline{P}_0(s)\,\frac{\partial}{\partial\lambda}$$

$$\left[\sum_{n=0}^{c-1}\frac{(N\rho+\phi)^n}{n!}+\frac{(N\rho+\phi)^c}{c!}\,{}_1F_1(1;\delta+1;N\beta\gamma+\psi)+\frac{(N\rho+\phi)^c}{c!}\frac{(N\beta\gamma+\psi)^{y-c+1}}{(\delta+1)_{y-c+1}}\,{}_2F_1(1,(N-1)+\psi/\beta\gamma,\delta+Y+2-c,\beta\gamma\right]$$

(3.1.20)

$$(I)\ \lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\Big[\sum_{n=0}^{c-1}\frac{(N\rho+\phi)^n}{n!}\Big]=\lambda\ \overline{P}_0(s)\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{c-1}\frac{(N\frac{\lambda}{\mu}+\frac{s}{\mu})^n}{n!}\right]$$

$$=\lambda\ \overline{P}_0(s)\,\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{c-1}(N\lambda+s)^n\frac{1}{n!\mu^n}\right]$$

$$=\lambda\ \overline{P}_0(s)\left[\sum_{n=0}^{c-1}n(N\lambda+s)^{n-1}\frac{1.N}{n!\mu^n}\right]$$

$$=\lambda\,\overline{P}_0(s)\sum_{n=0}^{c-1}\frac{(N\sigma+\eta)^{n-1}}{(n-1)!}\frac{N}{\mu}$$

$$=\overline{P}_0(s)\sum_{n=0}^{c-1}\frac{(N\sigma+\eta)^{n-1}}{(n-1)!}\frac{N\lambda}{\mu}$$

$$=\overline{P}_0(s)\sum_{n=0}^{c-1}\frac{(N\sigma+\eta)^{n-1}}{(n-1)!}N\sigma \quad \text{Where} \qquad \sigma=\lambda/\mu, \quad \eta=S/\mu$$

(II) $\lambda\ \overline{P}_0(s)\dfrac{\partial}{\partial\lambda}\left[\dfrac{(N\rho+\phi)^c}{c!}\,{}_1F_1(1;\delta+1;N\beta\gamma+\psi)\right]$

$$=\lambda\ \overline{P}_0(s)\ [{}_1F_1\,(1;\ \delta+1;N\beta\gamma+\psi)]\frac{\partial}{\partial\lambda}\frac{(N\rho+\phi)^c}{c!}+$$

$$\lambda\overline{P}_o(s)\frac{(N\rho+\phi)^c}{c!}\frac{\partial}{\partial\lambda}[\,{}_1F_1(1;\delta+1;N\beta\gamma+\psi)\,]$$

(II) a $\qquad \lambda\,\overline{P}_0(s)\dfrac{\partial}{\partial\lambda}\dfrac{(N\rho+\phi)^c}{c!}=\lambda\ \ \overline{P}_0(s)\dfrac{\partial}{\partial\lambda}\left[\dfrac{(N\frac{\lambda}{\mu}+\frac{s}{\mu})^c}{c!}\right]$

$$=\lambda\ \overline{P}_0(s)\frac{1}{\mu^c}\frac{\partial}{\partial\lambda}\frac{(N\lambda+S)^c}{c!}$$

$$=\lambda\,\overline{P}_0(s)\frac{N}{\mu^c}\frac{(N\lambda+S)^{c-1}}{c!} \qquad C=\lambda\overline{P}_o(s)\frac{N}{\mu^c}\frac{(N\lambda+s)^{c-1}}{(c-1)!}$$

$$=N.\frac{\lambda}{\mu}\,\overline{P}_0(s)\frac{(N\sigma+\eta)^{c-1}}{(c-1)!}$$

$$=N\sigma\ \overline{P}_0(s)\ \frac{(N\sigma+\eta)^{c-1}}{(c-1)!}$$

$$=\overline{P}_0(s)\ N\sigma\frac{(N\sigma+\eta)^{c-1}}{(c-1)!}$$

(II) b $\lambda\,\overline{P}_0(s)\,\dfrac{(N\rho+\phi)^c}{c!}\,\dfrac{\partial}{\partial\lambda}\,[{}_1F_1(1;\delta+1;N\beta\gamma+\psi)]$

$$\frac{(N\rho+\phi)^c}{c!}\lambda\,\overline{P}_0(s)\,\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{\infty}\frac{(1)_n(N\beta\gamma+\psi)^n}{(\delta+1)_n n!}\right]$$

$$\frac{(N\rho+\phi)^c}{c!}\lambda\,\overline{P}_0(s)\,\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{\infty}\frac{(1)_n(N\beta\frac{\lambda}{\alpha}+\frac{s}{\alpha})^n}{(c\frac{\mu}{\alpha}+1)_n n!}\right]$$

$$\frac{(N\rho+\phi)^c}{c!}\lambda\,\overline{P}_0(s)\left[\sum_{n=0}^{\infty}\frac{(1)_n n(N\beta\lambda+s)^{n-1}}{(\delta+1)_n n!}\frac{N\beta}{\alpha^n}\right]$$

$$\frac{(N\rho+\phi)^c}{c!}\,\frac{\lambda}{\alpha}\,\overline{P}_0(s)\sum_{n=0}^{\infty}\frac{(1)_n.(N\beta\gamma+\psi)^{n-1}}{(\delta+1)_n(n-1)!}N\beta$$

$$\frac{(N\rho+\phi)^c}{c!}\,N\beta\gamma\,\overline{P}_0(s)\sum_{n=1}^{\infty}\frac{(1)_n(N\beta\gamma+\psi)^{n-1}}{(\delta+1)(\delta+2)_{n-1}(n-1)!}$$

$$\frac{(N\rho+\phi)^c}{c!}\,N\beta\gamma\,\overline{P}_0(s)\sum_{n=0}^{\infty}\frac{(1)_{n+1}(N\beta\gamma+\psi)^n}{(\delta+1)(\delta+2)_n n!}$$

$$\frac{(N\rho+\phi)^c}{c!}\,\frac{N\beta\gamma}{\delta+1}\,\overline{P}_0(s)\sum_{n=0}^{\infty}\frac{(2)_n(N\beta\gamma+\psi)^n}{(\delta+2)_n n!}$$

$$\frac{(N\rho+\phi)^c}{c!}\,\frac{N\beta\gamma}{\delta+1}\,\overline{P}_0(s)\;{}_1F_1(2;\delta+2;N\beta\gamma+\psi)$$

Hence

$$\lambda\;\overline{P}_0(s)\,\frac{\partial}{\partial\lambda}\left[\frac{(N\rho+\phi)^c}{c!}\,{}_1F_1(1;\delta+1;N\beta\gamma+\psi)\right]=$$

$$N\sigma \; \frac{(N\sigma+\eta)^{c-1}}{(c-1)!} \, {}_1F_1(1;\delta+1;N\beta\gamma+\psi)\; \overline{P}_0(s) \; +$$

$$\frac{N\beta\gamma}{(\delta+1)c!}\,(N\rho+\phi)^c \; {}_1F_1(2;\delta+2;N\beta\gamma+\psi)\; \overline{P}_0(s)$$

(III) $\lambda\,\overline{P}_0(s)\;\dfrac{\partial}{\partial\lambda}\left[\dfrac{(N\rho+\phi)^c}{c!}\,\dfrac{(N\beta\gamma+\psi)^{y-c+1}}{(\delta+1)_{y-c+1}}\,{}_2F_1(1;(N-1)+(\dfrac{\psi}{\beta\lambda});\delta+y+2-c;\beta\gamma)\right]$

$$=\frac{(N\rho+\phi)^c}{c!}\;\frac{{}_2F_1(1,[(N-1)+\dfrac{\psi}{\beta\gamma}];\delta+y+2-c;\beta\gamma)}{(\delta+1)_{y-c+1}}\,\lambda\,\overline{P}_0(s)\;\frac{\partial}{\partial\lambda}\left[(N\beta\gamma+\psi)^{y-c+1}\right]+$$

$$\frac{(N\rho+\phi)^c}{c!}\;\frac{(N\beta\gamma+\psi)^{y-c+1}}{(\delta+1)_{y-c+1}}\;\frac{\partial}{\partial\lambda}\,{}_2F_1(1;[\,(N-1)+\frac{\psi}{\beta\lambda}\,];\delta+y+2-c;\beta\gamma)\lambda\overline{P}_o(s)$$

(III)a $\lambda\,\overline{P}_0(s)\;\dfrac{\partial}{\partial\lambda}\left[(N\beta\dfrac{\lambda}{\alpha}+\dfrac{s}{\alpha})^{y-c+1}\right]$

$$\lambda\,\overline{P}_0(s)\;\frac{(y-c+1)}{\alpha^{y-c+1}}(N\beta\lambda+s)^{y-c)}N\beta$$

$$=N\beta\frac{\lambda}{\alpha}\;\overline{P}_0(s)\;(y-c+1)\;(N\beta\frac{\lambda}{\alpha}+\frac{s}{\alpha})^{y-c}$$

$$=N\beta\gamma\;\overline{P}_0(s)\;(y-c+1)\;(N\beta\gamma+\psi)^{y-c}$$

(III)b $\lambda\,\overline{P}_0(s)\;\dfrac{\partial}{\partial\lambda}\left[\displaystyle\sum_{n=0}^{\infty}\dfrac{(1)_n[(N-1)+\dfrac{\psi}{\beta\gamma}]_n(\beta\gamma)^n}{(\delta+y+2-c)_n\,n!}\right]$

$$=\lambda\,\overline{P}_o(s)\sum_{n=0}^{\infty}\frac{(1)_n[(n-1)+\psi/\beta\gamma]_n}{(\delta+y+2-c)_n\,n!}\;\frac{\partial}{\partial\lambda}(\beta\lambda/\alpha)^n+$$

$$\lambda\;\overline{P}_0(s)\sum_{n=0}^{\infty}\frac{(1)_n(\beta\gamma)^n}{(\delta+y+2-c)_n\,n!}\,\frac{\partial}{\partial\lambda}[(N-1)+\frac{\psi}{\beta\gamma}]_n$$

$$=\lambda\,\overline{P}_o(s)\sum_{n=0}^{\infty}\frac{(1)_n[(N-1)+\psi/\beta\gamma]_n}{(\delta+y+2-c)_n\,n!}\,n\frac{1}{\alpha^n}(\beta\lambda)^{n-1}\beta+\lambda\,\overline{P}_o(s)\sum_{n=0}^{\infty}\frac{(1)_n(\beta\gamma)^n}{(\delta+y+2-c)_n\,n!}\left(-\frac{\psi}{\beta\lambda\gamma}\right)^n$$

$$=\overline{P}_0(s)\,\beta\gamma\sum_{n=0}^{\infty}\frac{(1)_n[(N-1)+\psi/\beta\gamma]_n(\beta\gamma)^{n-1}}{(\delta+y+2-c)_n(n-1)!}+\overline{P}_o(s)\sum_{n=0}^{\infty}\frac{(1)_n(-\psi/\lambda)^n\lambda}{(\delta+y+2-c)_n\,n!}$$

$$=\overline{P}_0(s)\,\beta\gamma\sum_{n=1}^{\infty}\frac{(1)_n[(N-1)+\psi/\beta\gamma]_n(\beta\gamma)^{n-1}}{(\delta+y+2-c)_n(n-1)!}+\overline{P}_0(s)\sum_{n=0}^{\infty}\frac{(1)_n(-\psi/\lambda)^n\lambda}{(\delta+y+2-c)_n\,n!}$$

$$=\overline{P}_0(s)\,\beta\gamma\sum_{n=0}^{\infty}\frac{(1)_{n+1}[(N-1)+\psi/\beta\gamma]_{n+1}(\beta\gamma)^n}{(\delta+y+2-c)_{n+1}\,n!}+\lambda\,\overline{P}_0(s)\sum_{n=0}^{\infty}\frac{(1)_n(-\psi/\lambda)^n}{(\delta+y+2-c)_n\,n!}$$

$$=\overline{P}_0(s)\,\beta\gamma\sum_{n=0}^{\infty}\frac{(2)_n[(N-1)+\psi/\beta\gamma][(N-2)+\psi/\beta\gamma]_n(\beta\gamma)^n}{(\delta+y+2-c)(\delta+y+3-c)_n\,n!}$$

$$+\lambda\,\overline{P}_0(s)\sum_{n=0}^{\infty}\frac{(1)_n(-\psi/\lambda)^n}{(\delta+Y+2-c)_n\,n!}$$

$$=\overline{P}_o(s)\,\frac{\beta\gamma[(N-1)+\psi/\beta\gamma]}{(\delta+y+2-c)}\,{}_2F_1(2,(N-2)+\psi/\beta\gamma;\delta+y+3-c;\beta\gamma)+$$

$$\lambda\,\overline{P}_0(s)\,{}_1F_1(1;\delta+y+2-c;-\psi/\lambda)$$

Hence,

$$L=\overline{P}_0(s)\Bigg[N\sigma\frac{(N\sigma+\eta)^{n-1}}{(n-1)!}+N\sigma\frac{(N\sigma+\eta)^{c-1}}{(c-1)!}\,{}_1F_1(1;\delta+1;N\beta\gamma+\psi)$$

$$+\frac{N\beta\gamma}{(\delta+1)C!}(N\rho+\phi)^c\,{}_1F_1(2;\delta+2;N\beta\gamma+\psi)+N\beta\gamma\frac{(Y-C+1)(N\beta\gamma+\psi)^{Y-C}}{(\delta+1)_{Y-C+1}}\,\frac{(N\rho+\phi)^c}{C!}$$

$${}_2F_1(1,(N-1)+\psi/\beta\gamma);\delta+Y+2-C;\beta\gamma)+\frac{(N\rho+\phi)^c}{C!}\,\frac{(N\beta\gamma+\psi)^{Y-C+1}}{(\delta+1)_{Y-C+1}}\,\frac{\beta\gamma[(N-1)+\psi/\beta\gamma]}{(\delta+Y+2-C)}$$

$${}_2F_1(2,(N-2)+\psi/\beta\gamma);\delta+Y+3-C;\beta\gamma)+\frac{(N\rho+\phi)^c}{C!}\,\frac{(N\beta\gamma+\psi)^{Y-C+1}}{(\delta+1)_{Y-C+1}}\lambda\,{}_1F_1(1;\delta+Y+2-C;-\psi/\lambda)\Bigg]\quad(3.1.21)$$

3.2 SPECIAL CASE

$P_0'(t) = -N\lambda\, P_0(t) + \mu P_1(t), \quad n = 0$ (3.2.1)

$P_n'(t) = -(N\lambda + n\,\mu)\, P_n(t) + N\lambda\, P_{n-1}(t) + (n+1)\,\mu P_{n+1}(t), \quad 1 \le n < C$ (3.2.2)

$P_c'(t) = -(N\beta\lambda + C\mu) P_c(t) + N\lambda\, P_{c-1}(t) + (C\mu + \alpha)\, P_{c+1}(t), \quad n = C$ (3.2.3)

$P_n'(t) = -(N\beta\lambda + C\mu + (n-C)\,\alpha)\, P_n(t) + N\beta\lambda\, P_{n-1}(t) +$

$[C\mu + (n+1-C)\,\alpha]\, P_{n+1}(t), \quad C+1 \le n < Y$ (3.2.4)

$P_n'(t) = -[(N+Y-n)\,\beta\lambda + c\mu + (n-c)\,\alpha]\, P_n(t) + (N+Y-n+1)\,\beta\lambda P_{n-1}(t) +$

$[C\mu + (n+1-C)\alpha]\, P_{n+1}(t), \quad Y+1 \le n < Y+K$ (3.2.5)

$P_{Y+K}'(t) = -[C\mu + (Y+K-C)\,\alpha]\, P_{Y+K}(t) + (N-K+1)\,\beta\lambda\, P_{Y+K-1}(t), \quad n = Y+K$

(3.2.6)

Let $\alpha = 0$, $\beta = 1$ and $n = k$ then the above equations are

$P_0'(t) = -K\lambda\, P_0(t) + \mu\, P_1(t) \qquad n=0$ (3.2.7)

$P_n'(t) = -(K\lambda + n\mu)\, P_n(t) + K\lambda\, P_{n-1}(t) + (n+1)\,\mu\, P_{n+1}(t), \quad 1 \le n < C$ (3.2.8)

$P_c'(t) = -(K\lambda + C\mu)\, P_c(t) + K\lambda\, P_{c-1}(t) + C\mu\, P_{c+1}(t), \quad n = c$ (3.2.9)

$P_n'(t) = -(K\lambda + c\mu)\, P_n(t) + K\lambda\, P_{n-1}(t) + c\mu\, P_{n+1}(t) \qquad c+1 \le n < Y$ (3.2.10)

$P_n'(t) = -[(K+Y-n)\lambda + c\mu] P_n(t) + (K+Y-n+1)\lambda P_{n-1}(t) + c\mu P_n(t)\; P_{n+1}(t) \quad Y+1 \le n < Y+K$ (3.2.11)

$P_{Y+K}'(t) = -[c\mu\, P_{Y+K}(t) + \lambda\, P_{Y+K-1}(t) \qquad n=Y+K$ (3.2.12)

From equation (3.2.7)

$P_0'(t) = -K\lambda \bar{P}_0(t) + \mu \bar{P}_1(t)$, n = 0

Taking laplace transform

$$S\,\bar{P}_0(s) - P(0) = -K\lambda\;\bar{P}_0(s) + \mu\;\bar{P}_1(s)$$

$$\mu\;\bar{P}_1(s) = \bar{P}_0(s)\,[K\lambda + s]$$

$$\bar{P}_1(s) = \left[\frac{K\lambda + s}{\mu}\right]\bar{P}_0(s)$$

$\bar{P}_1(s) = (K\rho + \phi)\;\bar{P}_0(s)$ Where $\lambda/\mu = \rho$, $S/\mu = \phi$

From equation (3.2.8)

$$P_n'(t) = -(K\lambda + n\mu)P_n(t) + K\lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$$

taking laplace transform

$$S\,\bar{P}_n(s) - P_n(0) = -(K\lambda + n\mu)\;\bar{P}_n(s) + K\lambda\;\bar{P}_{n-1}(s) + (n+1)\,\mu\;\bar{P}_{n+1}(s)$$

$$(n+1)\,\mu\;\bar{P}_{n+1}(s) = (K\lambda + n\mu + s)\;\bar{P}_n(s)\,K\lambda\;\bar{P}_{n-1}(s) - P_n(0)$$

$$(n+1)\,\mu\;\bar{P}_{n+1}(s) = (K\lambda + n\mu + s)\;\bar{P}_n(s) - K\lambda\;\bar{P}_{n-1}(s) - S\;\bar{P}_{n-1}(s)$$

$$(n+1)\,\mu\;\bar{P}_{n+1}(s) = (K\lambda + n\mu + s)\;\bar{P}_n(s) - (K\lambda + s)\;\bar{P}_{n-1}(s)$$

$$(n+1)\;\bar{P}_{n+1}(s) = (K\rho + n + \phi)\;\bar{P}_n(s) - (K\rho + \phi)\;\bar{P}_{n-1}(s)$$

Put n=1

$$2\;\bar{P}_2(s) = (K\rho + 1 + \phi)\;\bar{P}_1(s) - (K\rho + \phi)\;\bar{P}_0(s)$$

$$2\ \bar{P}_2(s) = (K\rho + 1 + \phi)\ (K\rho + \phi)\ \bar{P}_0(s) - (K\rho + \phi)\ \bar{P}_0(s)$$

$$= (K\rho + \phi)\ \bar{P}_0(s)\ [\ K\rho + \phi]$$

$$= (K\rho + \phi)(K\rho + \phi)\ \bar{P}_0(s)$$

$$\bar{P}_2(s) = \frac{(K\rho + \phi)^2}{2!}\ \bar{P}_0(s)$$

Put n =2

$$3\ \bar{P}_3(s) = (K\rho + 2 + \phi)\ \bar{P}_2(s) - (K\rho + \phi)\ \bar{P}_1(s)$$

$$= (K\rho + 2 + \phi)\ \frac{(K\rho + \phi)^2}{2!}\ \bar{P}_0(s) - (K\rho + \phi)(K\rho + \phi)\ \bar{P}_0(s)$$

$$3\ \bar{P}_3(s) = \frac{(K\rho + \phi)^2}{2!}\ [\ K\rho + \phi]\ \bar{P}_0(s)$$

$$\bar{P}_3(s) = \frac{(K\rho + \phi)^3}{3!}\ \bar{P}_0(s)$$

Similarly

$$\bar{P}_n(s) = \frac{(K\rho + \phi)^n}{n!}\ \bar{P}_0(s)\ ,\ 0 \le n \le c-1,\ \text{Where } \rho = \lambda/\mu\ ,\ \phi = S/\mu \qquad (3.2.13)$$

From equation (3.2.9)

$$P_c'(t) = -(K\lambda + c\mu)P_c(t) + K\lambda P_{c-1}(t) + c\mu P_{c+1}(t) \qquad n = c$$

Taking laplace transform

$$S\ \bar{P}_c(s) - P_c(0) = -(\ K\lambda + c\mu\ )\ \bar{P}_c(s) + K\lambda\ \ \bar{P}_{c-1}(s)\ + c\mu\ \ \bar{P}_{c+1}(s)$$

$$c\mu\ \ \bar{P}_{c+1}(s) = (\ K\lambda + c\mu + s)\ \ \bar{P}_c(s) - K\lambda\ \ \bar{P}_{c-1}(s) - P_c(0)$$

$$c\mu\ \ \bar{P}_{c+1}(s) = (\ K\lambda + c\mu + s)\ \ \bar{P}_c(s) - K\lambda\ \ \bar{P}_{c-1}(s) - S\ \bar{P}_{c-1}(s)$$

$$c\mu \ \bar{P}_{c+1}(s) = (K\lambda + c\mu + s) \ \bar{P}_c(s) - (K\lambda + s) \ \bar{P}_{c-1}(s)$$

$$c \ \bar{P}_{c+1}(s) = (K\rho + c + \phi) \ \bar{P}_c(s) - (K\rho + \phi) \ \bar{P}_{c-1}(s)$$

$$c \ \bar{P}_{c+1}(s) = (K\rho + c + \phi) \ \frac{(K\rho+\phi)^c}{c!} \ \bar{P}_0(s) - (K\rho + \phi) \frac{(K\rho+\phi)^{c-1}}{(c-1)!} \ \bar{P}_0(s)$$

$$c \ \bar{P}_{c+1}(s) = \frac{(K\rho+\phi)^c}{c!} \ (K\rho + c + \phi - c)$$

$$\bar{P}_{c+1}(s) = \frac{(K\rho+\phi)^{c+1}}{c!} \frac{1}{c}$$

From equation (3.2.10)

$$P^1{}_n(t) = - [K\lambda + c\mu) P_n(t) + K\lambda \ P_{n-1}(t) + c\mu \ P_{n+1}(t)$$

Taking laplace transform

$$S \ \bar{P}_n(s) - P_n(0) = - (K\lambda + c\mu) \ \bar{P}_n(s) + K\lambda \ \bar{P}_{n-1}(s) + c\mu \ \bar{P}_{n+1}(s)$$

$$c\mu \ \bar{P}_{n+1}(s) = (K\lambda + c\mu + s) \ \bar{P}_n(s) - K\lambda \ \bar{P}_{n-1}(s) - P_n(0)$$

$$c\mu \ \bar{P}_{n+1}(s) = (K\lambda + c\mu + s) \ \bar{P}_n(s) - K\lambda \ \bar{P}_{n-1}(s) - S \ \bar{P}_{n-1}(s)$$

$$c\mu \ \bar{P}_{n+1}(s) = (K\lambda + c\mu + s) \ \bar{P}_n(s) - (K\lambda + s) \ \bar{P}_{n-1}(s)$$

Put n = c+1

$$c\mu \ \bar{P}_{c+2}(s) = (K\lambda + c\mu + s) \ \bar{P}_{c+1}(s) - (K\lambda + s) \ \bar{P}_c(s)$$

$$c\mu \ \bar{P}_{c+2}(s) = (K\lambda + c\mu + s) \ \frac{(K\rho+\phi)^{c+1}}{c!c} \ \bar{P}_0(s) - (K\lambda + s) \ \frac{(K\rho+\phi)^c}{c!} \ \bar{P}_0(s)$$

$$c \ \bar{P}_{c+2}(s) = (K\rho + c + \phi) \ \frac{(K\rho+\phi)^{c+1}}{c!c} \ \bar{P}_0(s) - (K\rho + \phi) \ \frac{(K\rho+\phi)^c}{c!} \ \bar{P}_0(s)$$

$$=(K\rho + c + \phi)\frac{(K\rho+\phi)^{c+1}}{c!c}\ \overline{P}_0(s) - \frac{(K\rho+\phi)^{c+1}}{c!}\ \overline{P}_0(s)$$

$$= \frac{(K\rho+\phi)^{c+1}}{c!c}\ \overline{P}_0(s)\ [\ K\rho+\phi+c-c]$$

$$c\ \ \overline{P}_{c+2}(s) = \frac{(K\rho+\phi)^{c+2}}{c!c}\ \overline{P}_0(s)$$

$$\overline{P}_{c+2}(s) = \frac{(K\rho+\phi)^{c+2}}{c!c^2}\ \overline{P}_0(s)$$

$$\overline{P}_{c+2}(s) = \frac{(K\rho+\phi)^{c+2}}{c!}\left(\frac{1}{c}\right)^2\ \overline{P}_0(s)$$

Put $n = c+2$

$$c\mu\ \ \overline{P}_{c+3}(s) = (K\lambda + c\mu + s)\ \ \overline{P}_{c+2}(s) - (K\lambda + s)\ \ \overline{P}_{c+1}(s)$$

$$= (K\lambda + c\mu + s)\ \frac{(K\rho+\phi)^{c+2}}{c!}\left(\frac{1}{c}\right)^2\ \overline{P}_0(s) - (K\lambda + s)\ \frac{(K\rho+\phi)^{c+1}}{c!}\left(\frac{1}{c}\right)\ \ \overline{P}_0(s)$$

$$c\ \overline{P}_{c+3}(s) = (K\rho + c + \phi)\frac{(K\rho+\phi)^{c+2}}{c!}\left(\frac{1}{c}\right)^2\ \overline{P}_0(s) - (K\rho + \phi)\frac{(K\rho+\phi)^{c+1}}{c!}\left(\frac{1}{c}\right)\ \ \overline{P}_0(s)$$

$$= \frac{(K\rho+\phi)^{c+2}}{c!}\left(\frac{1}{c}\right)^2\ \overline{P}_0(s)\ [K\rho+\phi+c-c]$$

$$= \frac{(K\rho+\phi)^{c+2}}{c!}\left(\frac{1}{c}\right)^2\ \overline{P}_0(s)\ (K\rho+\phi)$$

$$\overline{P}_{c+3}(s) = \frac{(K\rho+\phi)^{c+3}}{c!}\left(\frac{1}{c}\right)^3\ \overline{P}_0(s)$$

Similarly

$$P_n = \bar{P}_c + (n-c) = \frac{(K\rho+\phi)^{n-c}}{c!}\left(\frac{1}{c}\right)^{n-c} \bar{P}_0(s), \quad C+1 \le n \le Y \qquad (3.2.14)$$

From equation (3.2.11)

$$P_n'(t) = -[(K+Y-n)\lambda + c\mu]P_n(t) + (K+Y-n+1)\lambda P_{n-1}(t) + c\mu P_{n+1}(t)$$

Taking laplace transform

$$S\,\bar{P}_n(s) - P_n(0) = -[(K+Y-n)\lambda + c\mu]\bar{P}_n(s) + (K+Y-n+1)\lambda \bar{P}_{n-1}(s) + c\mu \bar{P}_{n+1}(s)$$

$$C\mu\ \bar{P}_{n+1}(s) = [\,(K+Y-n)\,\lambda + c\,\mu + s]\ \bar{P}_n(s) - (\,K+Y-n+1)\,\lambda\ \bar{P}_{n-1}(s) - P_n(0)$$

$$C\mu\ \bar{P}_{n+1}(s) = [\,(K+Y-n)\,\lambda + c\,\mu + s]\ \bar{P}_n(s) - (\,K+Y-n+1)\,\lambda\ \bar{P}_{n-1}(s) - S\ \bar{P}_{n-1}(s)$$

$$C\mu\ \bar{P}_{n+1}(s) = [\,(K+Y-n)\,\lambda + c\,\mu + s]\ \bar{P}_n(s) - (\,K+Y-n+1)\,\lambda\ \bar{P}_{n-1}(s) - S\ \bar{P}_{n-1}(s)$$

$$C\ \bar{P}_{n+1}(s) = [\,(K+Y-n)\,\rho + c\ + \phi]\ \bar{P}_n(s) - (\,K+Y-n+1)\,\rho\ \bar{P}_{n-1}(s) - \phi\ \bar{P}_{n-1}(s)$$

Put $n = Y+1$

$$C\ \bar{P}_{Y+2}(s) = [\,(K+Y-Y-1)\,\rho + c\ + \phi]\ \bar{P}_{Y+1}(s) - (\,K)\,\rho\ \bar{P}_{Y+1-1}(s) - \phi\ \bar{P}_{Y+1-1}(s)$$

$$C\ \bar{P}_{Y+2}(s) = [\,(K-1)\,\rho + c\ + \phi]\ \bar{P}_{Y+1}(s) - (\,K\,)\,\rho\ \bar{P}_Y(s) - \phi\ \bar{P}_Y(s)$$

$$C\ \bar{P}_{Y+2}(s) = [\,(K-1)\,\rho + c\ + \phi]\ \bar{P}_{Y+1}(s) - (\,K\rho + \phi\,)\ \bar{P}_Y(s)$$

$$= [\,(K-1)\,\rho + c\ + \phi]\ \frac{(K\rho+\phi)^{Y+1-c}}{c!}\left(\frac{1}{c}\right)^{Y+1-c} \bar{P}_0(s) - (\,k\rho + \phi\,)\ \frac{(K\rho+\phi)^{Y-c}}{c!}\left(\frac{1}{c}\right)^{Y-c} \bar{P}_0(s)$$

$$C\ \bar{P}_{Y+2}(s) = \frac{(K\rho+\phi)^{Y+1-c}}{c!}\left(\frac{1}{c}\right)^{Y+1-c} \bar{P}_0(s)\ [(K-1)\,\rho + \phi + c - c]$$

$$= \frac{(K\rho+\phi)^{Y+1-c}}{c!}\left(\frac{1}{c}\right)^{Y+1-c} \bar{P}_0(s)\ [(K-1)\,\rho + \phi]$$

$$\bar{P}_{Y+2}(s) = \frac{(K\rho+\phi)^{Y-c}}{c!}\left(\frac{1}{c}\right)^{Y+1-c} (K\rho+\phi)\,[(K-1)\rho+\phi]\;\bar{P}_0(s)$$

$$\bar{P}_{Y+2}(s) = \frac{(K\rho+\phi)^{y-c}}{c!}\left(\frac{1}{c}\right)^{2}\left(\frac{1}{c}\right)^{Y-c} (K\rho+\phi)\,[(K-1)\rho+\phi]\;\bar{P}_0(s)$$

Put n = y + 2

$$C\;\bar{P}_{Y+3}(s) = [\,(K-2)\rho + c + \phi]\;\bar{P}_{Y+2}(s) - (K+Y-Y-2+1)\rho\;\bar{P}_{Y+1}(s) - \phi\;\bar{P}_{Y+1}(s)$$

$$C\;\bar{P}_{Y+3}(s) = [\,(K-2)\rho + c + \phi]\;\bar{P}_{Y+2}(s) - (K-1)\rho\;\bar{P}_{Y+1}(s) - \phi\;\bar{P}_{Y+1}(s)$$

$$= [\,(K-2)\rho + c + \phi]\;\bar{P}_{Y+2}(s) - [(K-1)\rho+\phi]\;\bar{P}_{Y+1}(s)$$

$$= [\,(K-2)\rho + c + \phi]\frac{(K\rho+\phi)^{Y-c}}{c!}\left(\frac{1}{c}\right)^{2}\left(\frac{1}{c}\right)^{Y-c} (K\rho+\phi)\,[(K-1)\rho+\phi]\;\bar{P}_0(s)\; -$$

$$[(K-1)\rho+\phi]\;\frac{(K\rho+\phi)^{Y+1-c}}{c!}\left(\frac{1}{c}\right)^{Y+1-c}\;\bar{P}_0(s)$$

$$= \frac{(K\rho+\phi)^{Y-c}}{c!}\left(\frac{1}{c}\right)^{2}\left(\frac{1}{c}\right)^{Y-c} (K\rho+\phi)\,[(K-1)\rho+\phi]\;\bar{P}_0(s)\,[(K-2)\rho+\phi+c-c]$$

$$C\;\bar{P}_{Y+3}(s) = \frac{(K\rho+\phi)^{Y-c}}{c!}\left(\frac{1}{c}\right)^{Y-c}\left(\frac{1}{c}\right)^{2} (K\rho+\phi)\,[(K-1)\rho+\phi][(K-2)\rho+\phi]\;\bar{P}_0(s)$$

$$\bar{P}_{Y+3}(s) = \frac{(K\rho+\phi)^{Y-c}}{c!}\left(\frac{1}{c}\right)^{Y-c}\left(\frac{1}{c}\right)^{3} (K\rho+\phi)\,[(K-1)\rho+\phi][(K-2)\rho+\phi]\;\bar{P}_0(s)$$

Hence

$$\bar{P}n = \bar{P}_{Y+(n-y)} =$$

$$= \frac{(K\rho+\phi)^{Y-c}}{c!}\left(\frac{1}{c}\right)^{Y-c}\left(\frac{1}{c}\right)^{n-Y} [K\rho+\phi]_{n-Y}\;\bar{P}_0(s)$$

$$\bar{P}_n(s) = \frac{(K\rho+\phi)^{Y-c}}{c!c^{Y-c}}\left(\frac{1}{c}\right)^{n-Y} [K\rho+\phi]_{n-Y}\ \bar{P}_0(s)$$

$$\bar{P}_n(s) = \frac{(K\rho+\phi)^{Y-c}}{c!c^{Y-c}}\left(\frac{1}{c}\right)^{n-Y} \rho^{n-Y}[K+\phi/\rho]_{n-Y}\ \bar{P}_0(s),\quad Y+1 \le n < Y+k \tag{3.2.15}$$

Thus $\bar{P}_n$ can be written as

$$\bar{P}_n = \begin{cases} \dfrac{(K\rho+\phi)^n}{n!}\ \bar{P}_0(s) & 0 \le n < c \\[2ex] \dfrac{(K\rho+\phi)^{n-c}}{c!}\left(\dfrac{1}{c}\right)^{n-Y}\ \bar{P}_0(s) & C+1 \le n < Y \\[2ex] \dfrac{(K\rho+\phi)^{Y-c}}{c!.c^{Y-c}}\left(\dfrac{1}{c}\right)^{n-Y} \rho^{n-Y}[K+\phi/\rho]_{n-Y}\ \bar{P}_0(s), & Y+1 \le n < K \end{cases} \tag{3.2.16}$$

Where $\rho = \lambda/\mu$, $\phi = S/\mu$

$$\bar{P}_0^{-1}(s) = \sum_{n=0}^{c-1}\frac{(K\rho+\phi)^n}{n!} + \sum_{n=c}^{Y}\frac{(K\rho+\phi)^{n-c}}{c!}\left(\frac{1}{c}\right)^{n-c} + \sum_{n=Y+1}^{Y+k}\frac{(K\rho+\phi)^{Y-c}}{c!c^{Y-c}}\left(\frac{\rho}{c}\right)^{n-y}\left[K+\frac{\phi}{\rho}\right]_{n-Y}$$

$$= \sum_{n=0}^{c-1}\frac{(K\rho+\phi)^n}{n!} + \frac{1}{c!}\sum_{n=c}^{Y}\left(\frac{K\rho+\phi}{c}\right)^{n-c} + \frac{(K\rho+\phi)^{Y-c}}{c!c^{Y-c}}\sum_{n=Y+1}^{Y+k}\left[(K)+\frac{\phi}{\rho}\right]_{n-Y}\left(\frac{\rho}{c}\right)^{n-Y} \tag{3.2.17}$$

$$\sum_{n=c}^{Y}\left(\frac{K\rho+\phi}{c}\right)^{n-c} \qquad n \Rightarrow n+c$$

$$\sum_{n=0}^{Y-c}\left(\frac{K\rho+\phi}{c}\right)^{n}\frac{n!}{n!} = \sum_{n=0}^{Y-c}\left(\frac{K\rho+\phi}{c}\right)^{n}\frac{(1)_n}{n!} = {}_1F_0\,(1; -; (K\rho+\phi)/c)$$

$$\sum_{n=Y+1}^{Y+k}\left[(K)+\frac{\phi}{\rho}\right]_{n-Y}\left(\frac{\rho}{c}\right)^{n-Y} \quad n \longrightarrow n+Y+1$$

$$\sum_{n=0}^{K-1}[(K)+\frac{\phi}{\rho}]_{n+1}\left(\frac{\rho}{c}\right)^{n+1}$$

$$\sum_{n=0}^{K-1}[K+\frac{\phi}{\rho}]_{n+1}\left(\frac{\rho}{c}\right)^{n+1}$$

$$\sum_{n=0}^{K-1}[K+\frac{\phi}{\rho}]\left((K-1)+\frac{\phi}{\rho}\right)_n\left(\frac{\rho}{c}\right)\left(\frac{\rho}{c}\right)^n$$

$$\left(\frac{K\rho+\phi}{c}\right)\sum_{n=0}^{K-1}\left(\frac{(K-1)+\frac{\phi}{\rho})_n\left(\frac{\rho}{c}\right)^n(1)_n}{n!}\right)$$

$$\left(\frac{K\rho+\phi}{c}\right)\ {}_2F_0\,(\,1,\,K\text{-}1+\frac{\phi}{\rho};\text{-};\frac{\rho}{c})$$

$$\overline{P}_0^{-1}(s)=\sum_{n=0}^{c-1}\frac{(K\rho+\phi)^n}{n!}+\frac{1}{c!}\,{}_1F_0\,(\,1,\text{-};(K\rho+\phi)\,/c)+\frac{(K\rho+\phi)^{Y-c}}{c!c^{Y-c}}\left(\frac{K\rho+\phi}{c}\right)$$

$${}_2F_0\,(\,1,(K\text{-}1)+\phi/\rho;\text{-};\frac{\rho}{c})$$

Hence

$$\overline{P}_0^{-1}(s)=\sum_{n=0}^{c-1}\frac{(K\rho+\phi)^n}{n!}+\frac{1}{c!}\,{}_1F_0\,(\,1,\text{--};(K\rho+\phi)\,/c)+\frac{(K\rho+\phi)^{Y-c+1}}{c!c^{Y-c+1}}\,{}_2F_0\,(\,1,(K\text{-}1)+\phi/\rho;\text{-};\frac{\rho}{c}) \qquad \textbf{(3.2.18)}$$

$$L=\lambda\ \ \overline{P}_0(s)\ \frac{\partial}{\partial\lambda}\ \overline{P}_0^{-1}(s)$$

$$L=\lambda\ \ \overline{P}_0(s)\ \frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{c-1}\frac{(K\rho+\phi)^n}{n!}+\frac{1}{c!}{}_1F_0(1;-;(K\rho+\phi)/c)+\frac{(K\rho+\phi)^{Y-c+1}}{c!c^{Y-c+1}}{}_2F_0(1,(K-1)+\frac{\phi}{\rho};-;\frac{\rho}{c})\right]$$

(I) $\quad\lambda\ \overline{P}_0(s)\ \frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{c-1}\frac{(K\rho+\phi)^n}{n!}\right]=\lambda\ \ \overline{P}_0(s)\frac{1}{n!}\ \frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{c-1}(K\frac{\lambda}{\mu}+\frac{s}{\mu})^n\right]$

$$= \lambda \ \overline{P}_0(s) \ \frac{1}{n!} \ \frac{\partial}{\partial \lambda} \left[\sum_{n=0}^{c-1} (\frac{K\lambda + s}{\mu})^n \right]$$

$$= \lambda \ \overline{P}_0(s) \ \frac{1}{n!} \left[\sum_{n=0}^{c-1} \frac{n(K\lambda + s)^{n-1}}{\mu^n} k \right]$$

$$= \sum_{n=0}^{c-1} (\frac{K\lambda + s}{\mu})^{n-1} \ \frac{K\lambda}{\mu} \ \ \frac{1}{(n-1)!} \ \overline{P}_0(s)$$

$$= \sum_{n=0}^{c-1} (K\rho + \phi)^{n-1} \, (K\rho) \ \frac{1}{(n-1)!} \ \overline{P}_0(s)$$

$$= \ \overline{P}_0(s) \ \sum_{n=0}^{c-1} \frac{(K\rho + \phi)^{n-1}}{(n-1)!} K\rho$$

(II) $\quad \lambda \ \overline{P}_0(s) \ \frac{\partial}{\partial \lambda} \left[\frac{1}{c!} {}_1F_0(1;-;(K\rho + \phi)/c \right]$

$$= \lambda \ \overline{P}_0(s) \ \frac{\partial}{\partial \lambda} \left[\frac{1}{c!} \sum_{n=0}^{\infty} \left(\frac{K\rho + \phi}{c} \right)^n \right]$$

$$= \lambda \ \overline{P}_0(s) \ \frac{\partial}{\partial \lambda} \left[\frac{1}{c!} \sum_{n=0}^{\infty} \left(\frac{K\lambda + s}{\mu c} \right)^n \right]$$

$$= \lambda \ \overline{P}_0(s) \left[\frac{1}{c!} \sum_{n=0}^{\infty} nK \frac{(K\lambda + s)^{n-1}}{\mu^n c^n} \right]$$

$$= \lambda \ \overline{P}_0(s) \left[\frac{1}{c!} \sum_{n=0}^{\infty} \frac{nK \left(\frac{(K\lambda + s)}{\mu} \right)^{n-1}}{\mu c^n} \right]$$

$$= \lambda \ \overline{P}_0(s) \left[\frac{1}{c!} \sum_{n=0}^{\infty} \frac{nK(K\rho + \phi)^{n-1}}{\mu c^n} \right]$$

$$= \bar{P}_0(s)\left[\frac{1}{c!}\sum_{n=0}^{\infty} n\frac{K\lambda}{\mu}\frac{(K\rho+\phi)^{n-1}}{c^n}\right]$$

$$= \bar{P}_0(s)\left[\frac{1}{c!}\sum_{n=0}^{\infty} nK\rho\frac{(K\rho+\phi)^{n-1}}{c^n}\right]$$

$$= \bar{P}_0(s)\left[\frac{1}{c!c}\sum_{n=0}^{\infty} nK\rho\left(\frac{K\rho+\phi}{c}\right)^{n-1}\right]$$

$$= \bar{P}_0(s)\,\frac{K\rho}{c!c}\sum_{n=0}^{\infty}\frac{(1)_n\left(\frac{K\rho+\phi}{c}\right)^{n-1} n}{n!}$$

$$= \bar{P}_0(s)\,\frac{K\rho}{c!c}\sum_{n=1}^{\infty}\frac{(1)_n\left(\frac{K\rho+\phi}{c}\right)^{n-1}}{(n-1)!}$$

$$= \bar{P}_0(s)\,\frac{K\rho}{c!c}\sum_{n=0}^{\infty}\frac{(1)_{n+1}\left(\frac{K\rho+\phi}{c}\right)^{n}}{n!}$$

$$= \bar{P}_0(s)\,\frac{K\rho}{c!c}\sum_{n=0}^{\infty}\frac{(2)_n\left(\frac{K\rho+\phi}{c}\right)^{n}}{n!}$$

$$= \bar{P}_0(s)\,\frac{K\rho}{c!c}\,{}_1F_0(2;-;((K\rho+\phi)/c))$$

(III) $\lambda\bar{P}_0(s)\dfrac{\partial}{\partial\lambda}\dfrac{(K\rho+\phi)^{Y-c+1}}{c!c^{Y-c+1}}\,{}_2F_0\left(1,(K-1)+\phi/\rho;-;\dfrac{\rho}{c}\right)$

$$=\frac{\lambda\bar{P}_0(s)\,{}_2F_0(1,(K-1)+\phi/\rho;-;\rho/c)}{c!c^{Y-c+1}}\frac{\partial}{\partial\lambda}(K\rho+\phi)^{Y-c+1}+\lambda\bar{P}_0(s)\frac{(K\rho+\phi)^{Y-c+1}}{c!c^{Y-c+1}}\frac{\partial}{\partial\lambda}$$

$$\left[{}_2F_0\left(1,(K-1)+\phi/\rho;-;\frac{\rho}{c}\right)\right]$$

(III)a $\lambda \frac{\partial}{\partial \lambda}(K\rho+\phi)^{Y-c+1} = \lambda \frac{\partial}{\partial \lambda}\left[K\frac{\lambda}{\mu}+\phi\right]^{Y-c+1}$

$$= (Y-c+1)\left[K\frac{\lambda}{\mu}+\phi\right]^{Y-c}\frac{K}{\mu}\lambda$$

$$= (Y-c+1)\ (K\rho+\phi)^{Y-c}\ K\rho$$

(III)b $\lambda \frac{\partial}{\partial \lambda}[{}_2F_0\,(\,1,(K-1)+\phi/\rho;-;\frac{\rho}{c})]$

$$= \lambda\frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty}\frac{(1)_n[(K-1)+\phi/\rho]_n\left(\frac{\rho}{c}\right)^n}{n!} = \lambda\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{\infty}\left[(K-1)+\frac{\phi\mu}{\lambda}\right]_n\left(\frac{\lambda}{\mu c}\right)^n\right]$$

$$= \lambda\left[\sum_{n=0}^{\infty}\left[(K-1)+\frac{\phi\mu}{\lambda}\right]_n\frac{\partial}{\partial\lambda}\left(\frac{\lambda}{\mu c}\right)^n + \left(\frac{\lambda}{\mu c}\right)^n\frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty}\left[(K-1)+\frac{\phi\mu}{\lambda}\right]_n\right]$$

$$= \lambda\left[\sum_{n=0}^{\infty}\left[(K-1)+\frac{\phi\mu}{\lambda}\right]_n n(\lambda)^{n-1}\frac{1}{(\mu c)^n} + \left(\frac{\lambda}{\mu c}\right)^n\sum_{n=0}^{\infty}\frac{\left(\frac{-\phi\mu}{\lambda}\right)^n(1)_n}{n!}\right]$$

$$= \sum_{n=0}^{\infty}\left[(K-1)+\frac{\phi\mu}{\lambda}\right]_n n\left(\frac{\lambda}{\mu c}\right)^n + \left(\frac{\lambda}{\mu c}\right)^n\lambda\sum_{n=0}^{\infty}\frac{\left(\frac{-\phi\mu}{\lambda}\right)^n(1)_n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{[(K-1)+\phi/\rho]_n n\left(\frac{\rho}{c}\right)^n(1)_n}{n!} + \left(\frac{\rho}{c}\right)^n \lambda\ {}_1F_0(1;-;-\phi\frac{\mu}{\lambda})$$

$$= \sum_{n=1}^{\infty}\frac{[(K-1)+\phi/\rho]_n\left(\frac{\rho}{c}\right)^n(1)_n}{(n-1)!} + \left(\frac{\rho}{c}\right)^n \lambda_1F_0(1;-;-\phi\frac{\mu}{\lambda})$$

$$=\left[(K-1)+\phi/\rho\right]\left(\frac{\rho}{c}\right)\sum_{n=0}^{\infty}\frac{\left[(K-2)+\phi/\rho\right]_n n\left(\frac{\rho}{c}\right)^n(2)_n}{n!}+\lambda\left(\frac{\rho}{c}\right)^n {}_1F_0(1;-;-\phi\frac{\mu}{\lambda})$$

$$=\left[(K-1)+\phi/\rho\right]\left(\frac{\rho}{c}\right){}_2F_0(2,(K-2)+\phi/\rho;-;\rho/c)+\lambda\left(\frac{\rho}{c}\right)^n {}_1F_0(1;-;-\frac{1}{\rho})$$

Hence

$$\mathrm{L}=\overline{P}_0(s)\left[\sum_{n=0}^{c-1}(K\rho+\phi)^{n-1}K\rho\frac{1}{(n-1)!}+\frac{K\rho}{c!c}{}_1F_0(2;-;(K\rho+\phi)/c)+\right.$$

$$\frac{(Y-c+1)K\rho(K\rho+\phi)^{Y-c}\,{}_2F_0(1,(K-1)+\frac{\phi}{\rho};-;\frac{\rho}{c})}{C!C^{Y-C+1}}+$$

$$\frac{(K\rho+\phi)^{Y-c+1}}{c!c^{Y-c+1}}\lambda\left(\frac{\rho}{c}\right)^n {}_2F_0(1,(K-1)+\frac{\phi}{\rho};-;\frac{\rho}{c})+$$

$$\frac{(K\rho+\phi)^{Y-c+1}}{c!c^{Y-c+1}}\lambda\left(\frac{\rho}{c}\right)^n {}_1F_0(1;-;\frac{-1}{\rho})+\frac{(K\rho+\phi)^{Y-c+1}}{c!c^{Y-c+1}}\left[(K-1)+\frac{\phi}{\rho}\right]\left(\frac{\rho}{c}\right){}_2F_0(2,(K-2)+\frac{\phi}{\rho};-;\frac{\rho}{c})$$

(3.2.19)

3.3 Case - II

This case treats the system of machine interference :

M/M/C/K/N with balking, reneging and spares Y<C and hence the birth-death coefficients are :

$$\lambda_n = \begin{cases} N\lambda & 0 \le n < Y \\ N+Y-n)\lambda & Y \le n < C \\ (N+Y-n)\beta\lambda & C \le n < Y+K \\ 0 & n \ge Y+K \end{cases}$$

and

$$\mu_n = \begin{cases} n\mu & 0 \le n < C \\ c\mu+(n-c)\alpha, & C+1 \le n < Y+K \end{cases}$$

Then as before the probability differential difference equation are :-

$$P_0'(t) = -N\lambda\, P_0(t) + \mu P_1(t) \qquad n=0 \qquad (3.3.1)$$

$$P_n'(t) = -(N\lambda + n\mu)\, P_n(t) + N\lambda\, P_{n-1}(t) + (n+1)\,\mu P_{n+1}(t), \qquad 1 \le n < Y \qquad (3.3.2)$$

$$P_n'(t) = -[(N+Y-n)\,\lambda + n\mu]P_n(t) + (N+Y-n+1)\,\lambda P_{n-1}(t) + (n+1)\,\mu P_{n+1}(t), \quad Y+1 \le n < C \qquad (3.3.3)$$

$$P_c'(t) = -[(N+Y-c)\,\beta\lambda + C\mu]\, P_c(t) + (N+Y-C+1)\,\lambda\, P_{c-1}(t) + (C\mu+\alpha)\, P_{c+1}(t) \qquad n=c \qquad (3.3.4)$$

$$P_n'(t) = -[(N+Y-n)\,\beta\lambda + C\mu + (n-C)\alpha]\, P_n(t) + (N+Y-n+1)\,\beta\lambda P_{n-1}(t) +$$

$$[C\mu + (n+1-C)\alpha]\, P_{n+1}(t), \quad C+1 \le n < Y+K \qquad (3.3.5)$$

$$P_{Y+K}'(t) = -[c\mu + (Y+K-c)\,\alpha]\, P_{Y+K}(t) + [N-K+1]\,\beta\lambda P_{Y+K-1}, \quad n = Y+K \qquad (3.3.6)$$

Taking laplace transformation then the differential equations are :

$$S\,\bar{P}_0(s) - P(0) = -N\lambda\,\bar{P}_0(s) + \mu\,\bar{P}_1(s), \qquad n=0 \qquad (3.3.7)$$

$$S\,\bar{P}_n(s) - P_n(0) = -(N\lambda + n\mu)\,\bar{P}_n(s) + N\lambda\,\bar{P}_{n-1}(s) + (n+1)\,\mu\,\bar{P}_{n+1}(s), \quad 1 \le n < Y \qquad (3.3.8)$$

$$S\,\bar{P}_n(s) - P_n(0) = -[(N+Y-n)\,\lambda + n\mu]\,\bar{P}_n(s) + (N+Y-n+1)\,\lambda\,\bar{P}_{n-1}(s) + (n+1)\mu\,\bar{P}_{n+1}(s)$$

$$Y+1 \leq n < C \qquad (3.3.9)$$

$$S\ \bar{P}_c(s) - P_c(0) = -[(N+Y-C)\beta\lambda + C\mu]\ \bar{P}_c(s) + (N+Y-C+1)\lambda\ \bar{P}_{c-1}(s) + [C\mu+\alpha]\ \bar{P}_{c+1}(s),\quad n = C \qquad (3.3.10)$$

$$S\ \bar{P}_n(s) - P_n(0) = -[(N+Y-n)\beta\lambda + C\mu + (n-C)\alpha]\ \bar{P}_n(s) + (N+Y-n+1)\beta\lambda\ \bar{P}_{n-1}(s) + [C\mu + (n+1-C)\alpha]\ \bar{P}_{n+1}(s),\quad C+1 \leq n < Y+K \qquad (3.3.11)$$

$$S\ \bar{P}_{Y+K}(s) - P_{Y+K}(0) = -[C\mu + (Y+K-C)\alpha]\ \bar{P}_{Y+K}(s) + (N-K+1)\beta\lambda\ \bar{P}_{Y+K-1}(s),\quad N = Y+K \qquad (3.3.12)$$

From equation (3.3.7)

$$S\ \bar{P}_0(s) - P(0) = -N\lambda\ \bar{P}_0(s) + \mu\ \bar{P}_1(s), \qquad n = 0 \qquad (3.3.7)$$

$$\mu\ \bar{P}_1(s) = (N\lambda + s)\ \bar{P}_0(s)$$

$$\bar{P}_1(s) = (N\rho + \phi)\ \bar{P}_0(s)$$

From equation (3.3.8)

$$S\ \bar{P}_n(s) - P_n(0) = -(N\lambda + n\mu)\ \bar{P}_n(s) + N\lambda\ \bar{P}_{n-1}(s) + (n+1)\mu\ \bar{P}_{n+1}(s),\quad 1 \leq n < Y \qquad (3.3.8)$$

$$S\ \bar{P}_n(s) - SP_{n-1}(s) = -(N\lambda + n\mu)\ \bar{P}_n(s) + N\lambda\bar{P}_{n-1}(s) + (n+1)\mu\ \bar{P}_{n+1}(s)$$

$$(n+1)\mu\bar{P}_{n+1}(s) = (N\lambda + n\mu + S)\ \bar{P}_n(s) - (N\lambda + S)\ \bar{P}_{n-1}(s)$$

$$(n+1)\bar{P}_{n+1}(s) = (N\rho + n + \phi)\ \bar{P}_n(s) - (N\rho + \phi)\ \bar{P}_{n-1}(s)$$

Put $n = 1$

$$2\bar{P}_2(s) = (N\rho + 1 + \phi)\ \bar{P}_1(s) - (N\rho + \phi)\ \bar{P}_0(s)$$

$$= (N\rho + \phi + 1)(N\rho + \phi)\ \bar{P}_0(s) - (N\rho + \phi)\ \bar{P}_0(s)$$

$$= (N\rho + \phi)\ \bar{P}_0(s)\ [N\rho + \phi]$$

$$\bar{P}_2(s) = \frac{(N\rho+\phi)^2}{2!}\bar{P}_0(s)$$

Put $\quad n = 2$

$$3\,\bar{P}_3(s) = (N\rho + 2 + \phi)\ \bar{P}_2(s) - (N\rho+\phi)\ \bar{P}_1(s)$$

$$= (N\rho + \phi + 2)\ \frac{(N\rho+\phi)^2}{2!}\bar{P}_0(s) - (N\rho + \phi)\ (N\rho + \phi)\ \bar{P}_0(s)$$

$$= \frac{(N\rho+\phi)^2}{2!}[N\rho+\phi]\bar{P}_0(S)$$

$$\bar{P}_3(s) = \frac{(N\rho+\phi)^3}{3!}\bar{P}_0(S)$$

Similarly

$$\bar{P}_n(s) = \frac{(N\rho+\phi)^n}{n!}\bar{P}_0(S) \qquad 0 \le n \le Y-1 \qquad (3.3.13)$$

From equation (3.3.9)

$$S\,\bar{P}_n(s) - P_n(0) = -(N+Y-n)\,\lambda + n\,\mu]\ \bar{P}_n(s) + (N+Y-n+1)\lambda\ \bar{P}_{n-1}(s) + (n+1)\,\mu\ \bar{P}_{n+1}(s)$$

$$S\,\bar{P}_n(s) - S\,\bar{P}_{n-1}(s) = -[(N+Y-n)\,\lambda + n\mu]\ \bar{P}_n(s) + (N+Y-n+1)\,\lambda\ \bar{P}_{n-1}(s) + (n+1)\,\mu\ \bar{P}_{n+1}(s)$$

$$(n+1)\,\mu\ \bar{P}_{n+1}(s) = [(N+Y-n)\,\lambda + n\mu + S]\ \bar{P}_n(s) - [(N+Y-n+1)\,\lambda + s]\ \bar{P}_{n-1}(s)$$

$$(n+1)\ \bar{P}_{n+1}(s) = [(N+Y-n)\,\rho + n + \phi]\ \bar{P}_n(s) - [(N+Y-n+1)\,\rho + \phi]\ \bar{P}_{n-1}(s)$$

Put $\quad n = Y+1$

$$(Y+2)\ \bar{P}_{Y+2}(s) = [(N+Y-Y-1)\,\rho + Y+1+\phi]\ \bar{P}_{Y+1}(s) - (N\rho+\phi)\ \bar{P}_Y(s)$$

$$= [(N-1)\,\rho + Y+1+\phi]\ \bar{P}_{Y+1}(s) - (N\rho+\phi)\ \bar{P}_Y(s)$$

$$= [(N-1)\,\rho + Y+1+\phi]\ \frac{(N\rho+\phi)^{Y+1}}{(Y+1)!}\ \bar{P}_0(s) - (N\rho+\phi)\frac{(N\rho+\phi)^Y}{Y!}\ \bar{P}_0(s)$$

$$= \frac{(N\rho+\phi)^{Y+1}}{(Y+1)!}[(N-1)\,\rho+\phi]\,\bar{P}_0(s)$$

$$\overline{P}_{Y+2}(s) = \frac{(N\rho+\phi)^Y}{Y!} \frac{(N\rho+\phi)[(N-1)\rho+\phi]}{(Y+1)(Y+2)} \overline{P}_0(s)$$

Put n = Y+2

$$(Y+3)\ \overline{P}_{y+3}(s) = [(N-2)\rho + Y+2+\phi]\ \overline{P}_{Y+2}(s) - [(N-1)\rho+\phi]\ \overline{P}_{Y+1}(s)$$

$$= [(N-2)\rho + Y+2+\phi] \frac{(N\rho+\phi)^Y (N\rho+\phi)[(N-1)\rho+\phi]}{Y!(Y+1)(Y+2)} \overline{P}_0(s) - [(N-1)\rho+\phi] \frac{(N\rho+\phi)^{Y+1}}{(Y+1)!} \overline{P}_0(s)$$

$$= \frac{(N\rho+\phi)^Y (N\rho+\phi)[(N-1)\rho+\phi]}{Y!(Y+1)(Y+2)} \overline{P}_0(s)\,[(N-2)\rho+\phi]$$

$$\overline{P}_{y+3}(s) = \frac{(N\rho+\phi)^Y (N\rho+\phi)[(N-1)\rho+\phi][(N-2)\rho+\phi]}{Y!(Y+1)(Y+2)(Y+3)} \overline{P}_0(s)$$

Similarly

$$\overline{P}_n = \overline{P}_{Y+(n-Y)} = \frac{(N\rho+\phi)Y}{Y!} \frac{(N\rho+\phi)_{n-Y}}{(Y+1)_{n-y}} \overline{P}_o(s)$$

$$= \frac{(N\rho+\phi)^Y \rho^{n-Y} (N+\frac{\phi}{\rho})_{n-Y}}{Y!(Y+1)_{n-Y}} \overline{P}_o(s), \qquad Y \le n \le C-1 \qquad (3.3.14)$$

From equation (3.3.10)

$$S\overline{P}_c(s) - P_c(0) = -[(N+Y-n)\beta\lambda + c\mu]\overline{P}_c(s) + (N+Y-c+1)\lambda\overline{P}_{c-1}(s) + (c\mu+\alpha)\overline{P}_{c+1}(s)$$

$$[c\mu+\alpha]\ \overline{P}_{c+1}(s) = [(N+Y-n)\beta\lambda + c\mu + S]\overline{P}_c(s) - (N+Y-c+1)\lambda\overline{P}_{c-1}(s) - S\overline{P}_{c-1}(s)$$

$$(\delta+1)\ \overline{P}_{c+1}(s) = [(N+Y-n)\beta\gamma + \delta + \psi]\overline{P}_c(s) - [(N+Y-C+1)\gamma + \psi]\overline{P}_{c-1}(s)$$

Where $\frac{c\mu}{\alpha} = \delta$, $\frac{\lambda}{\alpha} = \gamma$, $\frac{S}{\alpha} = \psi$

$$(\delta+1)\ \overline{P}_{c+1}(s) = [(N+Y-n)\beta\gamma + \delta + \psi] \frac{(N\rho+\phi)^Y}{Y!} \frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}} \overline{P}_o(s) -$$

$$\left[(N+Y-c+1)\gamma+\psi\right]\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y-1}}{(Y+1)_{c-1-Y}}\overline{P}_{o}(s) \qquad \textbf{(A)}$$

$$(N\rho+\phi)_{c-Y}=\left[(N\rho+\phi)(N(N-1)\rho+\phi)\ldots.(N-c+1+Y)\rho+\phi\right]$$

$N_n = N(N-1) \ldots\ldots(N-n+1)$

$(Y+1)_{C-Y} = (Y+C-Y)(Y+1)_{c-Y-1} = C(Y+1)_{c-Y-1}$

$$\left[(N+Y-c+1)\gamma+\psi\right]\frac{(N\rho+\phi)Y}{Y!}\frac{(N\rho+\phi)_{c-1-Y}}{c.(Y+1)_{c-1-Y}}$$

$$\frac{\mu}{\mu}\left[(N+Y-c+1)\frac{\lambda}{\alpha}+\frac{S}{\alpha}\right]\frac{(N\rho+\phi)Y}{Y!}\frac{(N\rho+\phi)_{c-1-Y}}{c.(Y+1)_{c-1-Y}}$$

$$\frac{\mu}{\alpha}\left[(N+Y-C+1)\rho+\phi\right](N\rho+\phi)_{c-1-Y}\frac{(N\rho+\phi)Y}{Y!}\frac{C}{(Y+1)_{c-Y}}$$

$$\frac{c\mu}{\alpha}\left[(N\rho+\phi)_{c-Y}\right]\frac{(N\rho+\phi)^{Y}}{Y!(Y+1)_{c-Y}}\overline{P}_{o}(s)$$

$$\delta\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\overline{P}_{o}(s)$$

Put in (A) we have

$$(\delta+1)\,\overline{P}_{c+1}(s)=\left[(N+Y-n)\beta\gamma+\delta+\psi\right]\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\overline{P}_{o}(s)-$$

$$\delta\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\overline{P}_{o}(s)$$

$$\overline{P}_{c+1}(s)=\left[\frac{(N+Y-n)\beta\gamma+\psi}{\delta+1}\right]\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\overline{P}_{o}(s)$$

From equation (3.3.11)

$$S\,\overline{P}_n(s) - S\,\overline{P}_{n-1}(s) = -\left[(N+Y-n)\beta\lambda + c\mu + (n-c)\alpha\right]\overline{P}_n(s) + (N+Y-n+1)\beta\lambda\,\overline{P}_{n-1}(s)$$

$$+[c\mu+(n+1-c)\alpha]\,\overline{P}_{n+1}(s)$$

$$\left[c\mu+(n+1-c)\alpha\right]\overline{P}_{n+1}(s) = \left[(N+Y-n)\beta\lambda + c\mu + (n-c)\alpha + S\right]\overline{P}_n(s) -$$

$$\left[(N+Y-n+1)\beta\lambda + S\right]\overline{P}_{n-1}(s)$$

$$\left[\delta+n+1-c\right]\overline{P}_{n+1}(s) = \left[(N+Y-n)\beta\gamma + \delta + n - c + \psi)\right]\overline{P}_n(s) - \left[(N+Y-n+1)\beta\gamma + \psi)\right]\overline{P}_{n-1}(s)$$

Put $n = c+1$

$$\left[\delta+2\right]\overline{P}_{c+2}(s) = \left[(N+Y-c-1)\beta\gamma + \delta + 1 + \psi\right]\overline{P}_{c+1}(s) - \left[(N+Y-c)\beta\gamma + \psi\right]\overline{P}_c(s)$$

$$\left[\delta+2\right]\overline{P}_{c+2}(s) = \left[(N+Y-c-1)\beta\gamma + \delta + 1 + \psi\right]\overline{P}_{c+1}(s) - \left[(N+Y-c)\beta\gamma + \psi\right]\overline{P}_c(s)$$

$$= \left[(N+Y-c-1)\beta\gamma + \delta + 1 + \psi\right]\left[\frac{(N+Y-c)\beta\gamma+\psi}{\delta+1}\right]\frac{(N\rho+\phi)^Y}{Y!}\;\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\;\overline{P}_0(s) -$$

$$\left[(N+Y-c)\beta\gamma+\psi\right]\frac{(N\rho+\phi)^Y}{Y!}\;\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\;\overline{P}_0(s)$$

$$= \left[(N+Y-c)\beta\gamma+\psi\right]\frac{(N\rho+\phi)^Y}{Y!}\;\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\;\overline{P}_0(s)\left[\frac{(N+Y-c-1)\beta\gamma+\delta+1+\psi)}{\delta+1}-1\right]$$

$$= \left[(N+Y-c)\beta\gamma+\psi\right]\frac{(N\rho+\phi)^Y}{Y!}\;\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\;\overline{P}_0(s)\left[\frac{(N+Y-c-1)\beta\gamma+\psi)}{\delta+1}\right]$$

$$\overline{P}_{c+2}(s) = \frac{(N\rho+\phi)^Y}{Y!}\;\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\;\frac{\left[(N+Y-c)\beta\gamma+\psi\right]\left[(N+Y-c-1)\beta\gamma+\psi\right]\overline{P}_0(s)}{(\delta+1)(\delta+2)}$$

Put $\quad n = c+2$

$$(\delta+3)\,\overline{P}_{c+3}(s) = \left[(N+Y-c-2)\beta\gamma + \delta + c + 2 - c + \psi\right]\overline{P}_{c+2}(s) - \left[(N+Y-c-1)\beta\gamma + \psi\right]\overline{P}_{c+1}(s)$$

$$= \left[(N+Y-c-2)\beta\gamma+\delta+2+\psi\right]\frac{(N\rho+\phi)^Y}{Y!}\;\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\;\frac{\left[(N+Y-c)\beta\gamma+\psi\right]\left[(N+Y-c-1)\beta\gamma+\psi\right]\overline{P}_0(s)}{(\delta+1)(\delta+2)}$$

$$-\left[(N+Y-c-1)\beta\gamma+\psi\right]\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\frac{\left[(N+Y-c)\beta\gamma+\psi\right]}{(\delta+1)}\overline{P}_0(s)$$

$=$

$$\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\frac{\left[(N+Y-c)\beta\gamma+\psi\right]\left[(N+Y-c-1)\beta\gamma+\psi\right]}{(\delta+1)}\overline{P}_0(s)\left[\frac{(N+Y-c-2)\beta\gamma+\delta+2+\psi}{(\delta+2)}-1\right]$$

$$\overline{P}_{c+3}(s)=\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\frac{\left[(N+Y-c)\beta\gamma+\psi\right]\left[(N+Y-c-1)\beta\gamma+\psi\right]\left[(N+Y-c-2)\beta\gamma+\psi\right]}{(\delta+1)(\delta+2)(\delta+3)}\overline{P}_0(s)$$

Similarly

$$\overline{P}_n(s)=\overline{P}_{c+(n-c)}(s)=\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\frac{\left[(N+Y-c)+\frac{\psi}{\beta\gamma}\right]_{n-c}(\beta\gamma)^{n-c}}{(\delta+1)_{n-c}}\overline{P}_0(s),\qquad C\leq n<Y+K$$

$$\overline{P}_n(s)=\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\frac{\left[(N+Y-c)+\frac{\psi}{\beta\gamma}\right]_{n-c}(\beta\gamma)^{n-c}}{(\delta+1)_{n-c}}\overline{P}_0(s),\ C\leq n<Y+K \qquad (3.3.15)$$

The empty system probability P_0, can be found from the boundary condition : $\sum_{n=0}^{Y+K}\overline{P}_n(s)=1$

$$\overline{P}_0^{-1}(s)=\sum_{n=0}^{Y-1}\frac{(N\rho+\phi)^{n}}{n!}+\frac{(N\rho+\phi)^{Y}}{Y!}\sum_{n=Y}^{c-1}\frac{\rho^{n-y}\left(N+\frac{\phi}{\rho}\right)_{n-y}}{(Y+1)_{n-Y}}+$$

$$\frac{(N\rho+\phi)^{Y}}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\sum_{n=c}^{Y+K}\frac{(\beta\gamma)^{n-c}\left[(N+Y-c)+\frac{\psi}{\beta\gamma}\right]_{n-c}}{(\delta+1)_{n-c}} \qquad (3.3.16)$$

(I) $\displaystyle\sum_{n=Y}^{c-1}\frac{\rho^{n-y}\left(N+\frac{\phi}{\rho}\right)_{n-y}}{(Y+1)_{n-Y}}$ $\qquad n\rightarrow n+Y$

$$\sum_{n=0}^{c-1-Y}\frac{\rho^{n}\left(N+\frac{\phi}{\rho}\right)_{n}(1)_{n}}{(Y+1)_{n}n!}={}_2F_1\left(1,N+\frac{\phi}{\rho};Y+1;\rho\right)$$

(II) $$\sum_{n=c}^{Y+K} \frac{(\beta\gamma)^{n-c}\left[(N+Y-c)+\frac{\psi}{\beta\gamma}\right]_{n-c}}{(\delta+1)_{n-c}} \qquad n \to n+c$$

$$=\sum_{n=0}^{Y+K-c} \frac{(\beta\gamma)^{n}\left[(N+Y-c)+\frac{\psi}{\beta\gamma}\right]_{n}(1)_n}{(\delta+1)_n n!} = {}_2F_1\left(1,(N+Y-c)+\frac{\psi}{\beta\gamma};\delta+1;\beta\gamma\right)$$

Hence

$$\overline{P}_0^{-1}(s)=\sum_{n=0}^{Y-1}\frac{(N\rho+\phi)^n}{n!}+\frac{(N\rho+\phi)^Y}{Y!}\,{}_2F_1\left(1,N+\frac{\phi}{\rho};Y+1;\rho\right)+\frac{(N\rho+\phi)^Y}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}$$

$${}_2F_1\left(1,(N+Y-c)+\frac{\psi}{\beta\gamma};\delta+1;\beta\gamma\right) \tag{3.3.17}$$

$$\mathrm{L} = \lambda\,\overline{P}_0^{-1}(s)\,\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{Y-1}\frac{(N\rho+\phi)^n}{n!}+\frac{(N\rho+\phi)^Y}{Y!}\,{}_2F_1\left(1,N+\frac{\phi}{\rho};Y+1;\rho\right)+\right.$$

$$\left.\frac{(N\rho+\phi)^Y}{Y!}\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\,{}_2F_1\left(1,(N+Y-c)+\frac{\psi}{\beta\gamma};\delta+1;\beta\gamma\right)\right] \tag{3.3.18}$$

(I)

$$\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\sum_{n=0}^{y-1}\frac{(N\rho+\phi)^n}{n!} = \lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{y-1}\frac{(N\frac{\lambda}{\mu}+\frac{s}{\mu})^n}{n!}\right]$$

$$= \lambda\overline{P}_0(s)\sum_{n=1}^{y-1}\frac{n(N\lambda+s)^{n-1}nN}{\mu^n n!}$$

$$=\sum_{n=1}^{y-1}N\rho\frac{(N\rho+\phi)^{n-1}}{(n-1)!}\,\overline{P}_0(s)$$

(II)

$$\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\left[\frac{(N\rho+\phi)^Y}{Y!}\,{}_2F_1(1,N+\frac{\phi}{\rho};Y+1;\rho)\right]$$

$$=\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\left[\frac{(N\rho+\phi)^Y}{Y!}\sum_{n=0}^{\infty}\frac{(1)_n(N+\phi/\rho)_n\rho^n}{(Y+1)_n n!}\right]$$

$$=\lambda\overline{P}_0(s)\,{}_2F_1(1,N+\frac{\phi}{\rho};Y+1;\rho)\frac{\partial}{\partial\lambda}\frac{(N\rho+\phi)^Y}{Y!}+\frac{(N\rho+\phi)^Y}{Y!}\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{\infty}\frac{(1)_n(N+\phi/\rho)_n\rho^n}{(Y+1)_n n!}\right]$$

(II)a $$\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\frac{(N\rho+\phi)^Y}{Y!}=\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\left[\frac{(N\lambda+s)^Y}{\mu^Y Y!}\right]$$

$$=\frac{\lambda\overline{P}_0(s)Y(N\lambda+s)^{Y-1}}{\mu^Y Y!}N$$

$$=\mathrm{N}\rho\frac{(N\rho+\phi)^{Y-1}}{(Y-1)!}=\mathrm{N}\rho\frac{(N\rho+\phi)^{Y-1}}{(Y-1)!}$$

(II)b $$\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{\infty}\frac{(1)_n(N+\phi/\rho)_n\rho^n}{(Y+1)_n n!}\right]$$

$$=\lambda\overline{P}_0(s)\left[\sum_{n=0}^{\infty}\frac{(1)_n(N+\phi/\rho)_n}{(Y+1)_n n!}\frac{\partial}{\partial\lambda}(\rho^n)+\sum_{n=0}^{\infty}\frac{(1)_n\rho^n}{(Y+1)_n n!}\frac{\partial}{\partial\lambda}\left(N+\frac{\phi}{\rho}\right)_n\right]$$

$$=\sum_{n=1}^{\infty}\frac{(1)_n(N+\phi/\rho)_n\rho^n}{(y+1)_n(n-1)!}+\lambda\sum_{n=0}^{\infty}\frac{(1)_n(-\frac{\phi}{\lambda})^n}{(Y+1)_n n!}$$

$$=\sum_{n=0}^{\infty}\frac{(1)_{n+1}(N+\phi/\rho)_{n+1}\rho^{n+1}}{(Y+1)_{n+1}n!}+\lambda\sum_{n=0}^{\infty}\frac{(1)_n(-\frac{\phi}{\lambda})^n}{(Y+1)_n n!}$$

$$=\sum_{n=0}^{\infty}\frac{(2)_n(N+\frac{\phi}{\rho})\ ((N-1)+\frac{\phi}{\rho})_n\rho\rho^n}{(Y+1)(Y+2)_n n!}+\lambda\sum_{n=0}^{\infty}\frac{(1)_n(-\frac{\phi}{\lambda})^n}{(Y+1)_n n!}$$

$$=\frac{(N\rho+\phi)}{(Y+1)}\sum_{n=0}^{\infty}\frac{(2)_n[(N-1)+\frac{\phi}{\rho}]_n\rho^n}{(Y+2)_n n!}+\lambda\sum_{n=0}^{\infty}\frac{(1)_n(-\frac{\phi}{\lambda})^n}{(Y+1)_n n!}$$

$$=\frac{(N\rho+\phi)}{(Y+1)}\,{}_2F_1(2,(N-1)+\phi/\rho;Y+2;\rho)+\lambda\ \ {}_1F_1(1;Y+1;-\phi/\lambda)$$

Hence

$$\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\left\{\frac{(N\rho+\phi)^Y}{Y!}{}_2F_1(1,N+\phi/\rho;Y+1;\rho\right\}=$$

$${}_2F_1(1,N+\phi/\rho;Y+1;\rho)\,\mathrm{N}\rho\frac{(N\rho+\phi)^{Y-1}}{(Y-1)!}+$$

$$\frac{(N\rho+\phi)^Y}{Y!}\left[\frac{(N\rho+\phi)}{(Y+1)}\,{}_2F_1(2,(N-1)+\phi/\rho;Y+2;\rho)+\lambda{}_1F_1(1;Y+1;-\phi/\lambda)\right]$$

(III) $$\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\left[\frac{(N\rho+\phi)^Y(N\rho+\phi)_{c-Y}}{Y!(Y+1)_{c-Y}}{}_2F_1(1,N+Y-c+\psi/\beta\gamma;\delta+1;\beta\gamma)\right]$$

$$=\lambda\overline{P}_0(s)\left[\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}{}_2F_1(1,N+Y-c+\psi/\beta\gamma;\delta+1;\beta\gamma)\right]$$

$$\frac{\partial}{\partial\lambda}\left[\frac{(N\rho+\phi)^Y}{Y!}\right]+\lambda\overline{P}_0(s)\left[\frac{(N\rho+\phi)^Y(N\rho+\phi)_{c-Y}}{Y!(Y+1)_{c-Y}}\frac{\partial}{\partial\lambda}{}_2F_1(1,N+Y-c+\psi/\beta\gamma;\delta+1;\beta\gamma)\right]$$

$$=\overline{P}_0(s)\left[\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}{}_2F_1(1,N+Y-c+\psi/\beta\gamma;\delta+1;\beta\gamma)N\rho\frac{(N\rho+\phi)^{Y-1}}{(Y-1)!}\right]+$$

$$\lambda\overline{P}_0(s)\left[\frac{(N\rho+\phi)^Y(N\rho+\phi)_{c-Y}}{Y!(Y+1)_{c-Y}}\frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty}\frac{(1)_n(N+Y-c+\psi/\beta\gamma)_n(\beta\gamma)^n}{(\delta+1)_n n!}\right]$$

$$\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\left[\sum_{n=0}^{\infty}\frac{(1)_n(N+Y-c+\psi/\beta\gamma)_n(\beta\gamma)^n}{(\delta+1)_n n!}\right]$$

$$=\lambda\overline{P}_0(s)\sum_{n=0}^{\infty}\frac{(1)_n(N+Y-c+\psi/\beta\gamma)_n}{(\delta+1)n!}\frac{\partial}{\partial\lambda}(\beta\gamma)^n+\lambda\overline{P}_0(s)\sum_{n=0}^{\infty}\frac{(1)_n(\beta\gamma)^n}{(\delta+1)_n n!}\frac{\partial}{\partial\lambda}(N+Y-c+\psi/\beta\gamma)_n$$

$$= \lambda \overline{P}_0(s) \sum_{n=0}^{\infty} \frac{(1)_n (N+y-c+\psi/\beta\gamma)_n}{(\delta+1)_n n!} n((\beta\lambda)^{n-1}) \frac{\beta}{\alpha^n} + \lambda \sum_{n=0}^{\infty} \frac{(1)_n (\beta\gamma)^n}{(\delta+1)_n n!} \left(-\frac{\psi}{\beta\lambda\gamma} \right)^n$$

$$= \sum_{n=1}^{\infty} \frac{(1)_n (N+Y-c+\psi/\beta\gamma)_n \beta\gamma}{(\delta+1)_n (n-1)!} (\beta\gamma)^{n-1} + \lambda \sum_{n=0}^{\infty} \frac{(1)_n \left(-\frac{\psi}{\lambda} \right)^n}{(\delta+1)_n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(1)_{n+1} (N+y-c+\psi/\beta\gamma)_{n+1} (\beta\gamma)^{n+1}}{(\delta+1)_{n+1} n!} + \lambda_1 F_1(1;\delta+1;-\psi/\lambda)$$

$$= \frac{(N+Y-c+\psi/\beta\gamma)\beta\gamma}{(\delta+1)} \sum_{n=0}^{\infty} \frac{(2)_n (N+Y-c-1+\psi/\beta\gamma)_n}{(\delta+2)_n n!} (\beta\gamma)^n + \lambda_1 F_1(1;\delta+1;-\psi/\lambda)$$

$$= \frac{(N+Y-c+\psi/\beta\gamma)}{(\delta+1)} {}_2F_1(2, N-1+Y-c+\psi/\beta\gamma; \delta+2; \beta\gamma) + \lambda_1 F_1(1;\delta+1;-\psi/\lambda)$$

Hence

$$\lambda \overline{P}_0(s) \frac{\partial}{\partial\lambda} \left[\frac{(N\rho+\phi)^Y (N\rho+\phi)_{c-Y}}{Y!(Y+1)_{c-Y}} {}_2F_1(1, N+Y-c+\frac{\psi}{\beta\gamma}; \delta+1; \beta\gamma) \right]$$

$$= \overline{P}_0(s) \Bigg[\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}} {}_2F_1(1, N+Y-c+\psi/\beta\gamma; \delta+1; \beta\gamma) \; N\rho \frac{(N\rho+\phi)^{Y-1}}{(Y-1)!}$$

$$+ \frac{(N\rho+\phi)^Y (N\rho+\phi)_{c-Y}}{Y!(Y+1)_{c-Y}} \frac{(N+Y-c+\psi/\beta\gamma)}{\delta+1} {}_2F_1(2, N-1+Y-c+\psi/\beta\gamma; \delta+2; \beta\gamma)$$

$$+ \frac{(N\rho+\phi)^Y}{Y!} \frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}} \lambda_1 F_1(1;\delta+1;-\psi/\lambda) \Bigg]$$

Hence

$$\mathrm{L} = \overline{P}_0(s) \Bigg[\sum_{n=1}^{Y-1} N\rho \frac{(N\rho+\phi)^{n-1}}{(n-1)!} + N\rho \frac{(N\rho+\phi)^{Y-1}}{(Y-1)!} {}_2F_1(1, N+\phi/\rho; Y+1; \rho)$$

$$+ \frac{(N\rho+\phi)^{Y+1}}{(Y+1)!} {}_2F_1(2, (N-1)+\phi/\rho; Y+2; \rho) + \frac{(N\rho+\phi)^Y}{Y!} \lambda_1 F_1(1, Y+1; -; -\phi/\lambda)$$

$$+\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\,N\rho\frac{(N\rho+\phi)^{Y-1}}{(Y-1)!}\;{}_2F_1(1,N+Y-c+\frac{\psi}{\beta\gamma};\delta+1;\beta\gamma)$$

$$+\;\frac{(N\rho+\phi)^{Y}}{Y!}\,\frac{(N\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\,\frac{(N+Y-c+\psi/\beta\gamma)}{\delta+1}\;{}_2F_1(2,N-1+Y-c+\frac{\psi}{\beta\gamma};\delta+2;\beta\gamma)$$

$$+\frac{(N\rho+\phi)^{Y}}{Y!}\,\frac{(N\rho+\phi)_{c-Y}\,\lambda}{(Y+1)_{c-Y}}\;{}_1F_1(1;\delta+1;-\frac{\psi}{\lambda})\Bigg] \qquad (3.3.19)$$

3.4 SPECIAL CASES

$P_0'(t) = -N\lambda\, P_0(t) + \mu\, P_1(t) \qquad n=0 \qquad (3.4.1)$

$P_n'(t) = -(N\lambda + n\mu)\, P_n(t) + N\lambda\, P_{n-1}(t) + (n+1)\,\mu\, P_{n+1}(t), \quad 1 \le n < Y \qquad (3.4.2)$

$P_n'(t) = -[(N+Y-n)\,\lambda + n\mu]\, P_n(t) + (N+Y-n+1)\,\lambda P_{n-1}(t) + (n+1)\,\mu P_{n+1}(t)$

$Y+1 \le n < c \qquad (3.4.3)$

$P_c'(t) = -[(N+Y-n)\,\beta\lambda + c\mu]\, P_c(t) + (N+Y-c+1)\,\lambda P_{c-1}(t) + (c\mu + \alpha)\, P_{c+1}(t), \ n = c$

$(3.4.4)$

$P_n'(t) = -[(N+Y-n)\,\beta\lambda + c\mu + (n-c)\,\alpha]\, P_n(t) + (N+Y-n+1)\,\beta\lambda\, P_{n-1}(t)$

$+[C\mu + (n+1-c)\alpha]\, P_{n+1}(t), \ c+1 \le n < Y+K \quad (3.4.5)$

$P_{y+k}'(t) = -[c\mu + (Y+K-c)\,\alpha]\, P_{Y+k}(t) + [N-K+1]\,\beta\lambda\, P_{Y+k-1}(t), \ n=Y+K \quad (3.4.6)$

Let $\alpha = 0$, $\beta = 1$ and N=K then the above equations are :-

$P_0'(t) = -K\lambda\, P_0(t) + \mu\, P_1(t) \qquad n=0 \qquad (3.4.7)$

$P_n'(t) = -(K\lambda + n\mu)\, P_n(t) + K\lambda\, P_{n-1}(t) + (n+1)\,\mu\, P_{n+1}(t), \quad 1 \le n < Y \qquad (3.4.8)$

$P_n'(t) = -[(K+Y-n)\,\lambda + n\mu]\, P_n(t) + (K+Y-n+1)\,\lambda P_{n-1}(t) + (n+1)\,\mu P_{n+1}(t)$

$Y+1 \le n < C \qquad (3.4.9)$

$P_c'(t) = -[(K+Y-n)\,\beta\lambda + c\mu]\, P_c(t) + (K+Y-c+1)\,\lambda P_{c-1}(t) + c\mu\, P_{c+1}(t), \ n = c$

$(3.4.10)$

$P_n'(t) = -[(K+Y-n)\,\beta\lambda + c\mu]\, P_n(t) + (K+Y-n+1)\,\beta\lambda\, P_{n-1}(t) + C\mu\, P_{n+1}(t),$

$$c+1 \le n < Y+K \qquad (3.4.11)$$

$$P'_{y+k}(t) = -c\mu\, P_{Y+k}(t) + (N-K+1)\,\lambda\, P_{Y+k-1}(t), \qquad n=Y+K$$

(3.4.12)

From equation (3.4.7)

$$P'_0(t) = -K\lambda\, P_0(t) + \mu P_1(t) \qquad n=0$$

Taking laplace transform

$$S\,\overline{P}_0(s) - P(o) = -K\lambda\ \overline{P}_0(s) + \mu\,\overline{P}_1(s)$$

$$\mu\,\overline{P}_1(s) = \overline{P}_0(s)\,[K\lambda+s]$$

$$\overline{P}_1(s) = (K\rho+\phi)\ \overline{P}_0(s) \text{ where } \rho=\lambda/\mu, \qquad \phi=S/\mu$$

From equation (3.4.8)

$$P'_n(t) = -(K\lambda + n\mu)\, P_n(t) + K\lambda\, P_{n-1}(t) + (n+1)\,\mu\, P_{n+1}(t), \qquad 1 \le n < Y$$

Taking laplace transform

$$\overline{P}_n(s) - P_n(0) = -(K\lambda + n\mu)\ \overline{P}_n(s) + K\lambda\,\overline{P}_{n-1}(s) + (n+1)\,\mu\,\overline{P}_{n+1}(s)$$

$$(n+1)\,\mu\,\overline{P}_{n+1}(s) = (K\lambda + n\mu + s)\ \overline{P}_n(s) - K\lambda\,\overline{P}_{n-1}(s) - P_n(0)$$

$$(n+1)\,\mu\,\overline{P}_{n+1}(s) = (K\lambda + n\mu + s)\ \overline{P}_n(s) - (K\lambda+s)\ \overline{P}_{n-1}(s)$$

$$(n+1)\ \overline{P}_{n+1}(s) = (K\rho+n+\phi)\ \overline{P}_n(s) - (K\rho+\phi)\ \overline{P}_{n-1}(s)$$

Put n =1

$$2\ \overline{P}_2(s) = (K\rho+1+\phi)\ \overline{P}_1(s) - (K\rho+\phi)\ \overline{P}_0(s)$$

$$= (K\rho+1+\phi)\ (K\rho+\phi)\ \overline{P}_0(s) - (K\rho+\phi)\,\overline{P}_0(s)$$

$2\ \overline{P}_2(s) = (\,K\rho+\phi)\ \overline{P}_0(s)\ [K\rho+\phi+1-1]$

$$\overline{P}_2(s) \ = \frac{(K\rho+\phi)^2}{2!}\ \overline{P}_0(s)$$

Put n =2

$3\,\overline{P}_3(s) = (\,K\rho+2+\phi)\ \ \overline{P}_2(s) - (\,K\rho+\phi)\,\overline{P}_1(s)$

$$= (\,K\rho+2+\phi)\ \ \frac{(K\rho+\phi)^2}{2!}\ \overline{P}_0(s) - (\,K\rho+\phi)(\,K\rho+\phi)\,\overline{P}_0(s)$$

$$= \frac{(K\rho+\phi)^2}{2!}\ [K\rho+\phi+2-2]\ \overline{P}_0(s)$$

$$\overline{P}_3(s) \ = \frac{(K\rho+\phi)^3}{3!}\ \overline{P}_0(s)$$

Similarly,

$$\overline{P}_n(s) \ = \frac{(K\rho+\phi)^n}{n!}\ \overline{P}_0(s), \qquad 0 \le n \le Y-1 \tag{3.4.13}$$

From equation (3.4.9)

$P_n'(t) = -[(K+Y-n)\,\lambda + n\mu]\ P_n(t) + (K+Y-n+1)\,\lambda P_{n-1}(t) + (n+1)\,\mu\, P_{n+1}(t)$,

$Y+1 \le n < c$

Taking laplace transform

$S\,\overline{P}_n(s) - P_n(0) = -[(K+Y-n)\,\lambda + n\mu]\ \ \overline{P}_n(s) + (\,K+Y-n+1)\,\lambda\ \overline{P}_{n-1}(s)\ + (n+1)\,\mu$

$\overline{P}_{n+1}(s)$

$(n+1)\,\mu\ \overline{P}_{n+1}(s) = [(K+Y-n)\lambda + n\mu + s]\ \overline{P}_n(s) - (K+Y-n+1)\,\lambda\ \overline{P}_{n-1}(s) - P_n(0)$

$(n+1)\,\mu\ \overline{P}_{n+1}(s) = [(K+Y-n)\,\lambda + n\,\mu + s]\ \overline{P}_n(s) - [(K+Y-n+1)\,\lambda + s]\ \overline{P}_{n-1}(s)$

Put n=Y+1

$(Y+2)\ \mu\ \overline{P}_{Y+2}(s) = [(K+Y-Y-1)\ \lambda + (Y+1)\ \mu + S]\ \overline{P}_{Y+1}(s) - [(K+Y-Y-1+1)\ \lambda + S]$

$\overline{P}_{Y+1-1}(s)$

$$(Y+2)\ \overline{P}_{Y+2}(s) = [(K-1)\ \rho + (Y+1) + \phi]\ \overline{P}_{Y+1}(s) - (K\rho + \phi)\ \ \overline{P}_{Y}(s)$$

$$(Y+2)\ \overline{P}_{Y+2}(s) = [(K-1)\ \rho + Y + 1 + \phi]\ \frac{(K\rho+\phi)^{Y+1}}{(Y+1)!}\ \overline{P}_0(s) - (K\rho + \phi)\ \ \frac{(K\rho+\phi)^{Y}}{Y!}\ \overline{P}_0(s)$$

$$= \frac{(K\rho+\phi)^{Y+1}}{(Y+1)!}[(K-1)\ \rho + Y + 1 + \phi - Y - 1]\ \overline{P}_0(s)$$

$$\overline{P}_{Y+2}(s) = \frac{(K\rho+\phi)^{Y}}{Y!}\ \frac{(K\rho+\phi)((K-1)\rho+\phi)}{(Y+1)(Y+2)}\ \overline{P}_0(s)$$

Put n = Y + 2

$(Y+3)\ \overline{P}_{Y+3}(s) = [(K+Y-Y-2)\ \rho + Y + 2 + \phi]\ \overline{P}_{Y+2}(s) - [(K+Y-Y-2+1)\ \rho +$

$\phi]\ \overline{P}_{Y+1}(s)$

$$= [(K-2)\ \rho + Y + 2 + \phi]\ \ \overline{P}_{Y+2}(s)\ - [(K-1)\ \rho + \phi]\ \overline{P}_{Y+1}(s)$$

$$= [(K-2)\ \rho + Y + 2 + \phi]\ \frac{(K\rho+\phi)^{Y}}{Y!}\ \frac{(K\rho+\phi)((K-1)\rho+\phi)}{(Y+1)(Y+2)}\ \overline{P}_0(s)$$

$$-[(K-1)\ \rho + \phi]\ \frac{(K\rho+\phi)^{Y+1}}{(Y+1)!}\ \overline{P}_0(s)$$

$$= \frac{(K\rho+\phi)^{Y}}{Y!}\ \frac{(K\rho+\phi)((K-1)\rho+\phi)}{(Y+1)(Y+2)}\ \overline{P}_0(s)[(k-2)\ \rho + Y + 2 + \phi - Y - 2]$$

$$\overline{P}_{Y+3}(s) \quad = \frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)((K-1)\rho+\phi)((K-2)\rho+\phi)}{(Y+1)(Y+2)(Y+3)}\overline{P}_0(s)$$

$$\overline{P}_n = \overline{P}_{Y+(n-Y)} = \frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{n-Y}}{(Y+1)_{n-Y}}\overline{P}_0(s)$$

$$Y \le n < c-1$$

$$\overline{P}_n = \frac{(K\rho+\phi)^Y}{Y!}\frac{\rho^{n-Y}(K+\frac{\phi}{\rho})_{n-Y}}{(Y+1)_{n-Y}}\overline{P}_0(s) \qquad (3.4.14)$$

From equation (3.4.10)

$P_c^1(t) = -[(K+Y-n)\,\lambda + c\mu\,]\,P_c(t) + (K+Y-c+1)\,\lambda\,P_{c-1}(t) + c\mu\ \ P_{c+1}(t)\ \ n=c$

Taking laplace transform

$S\,\overline{P}_c(s) - P_c(0) = -[(K+Y-n)\lambda + c\mu\,]\ \overline{P}_c(s) + (K+Y-c+1)\,\lambda\ \overline{P}_{c-1}(s) + c\mu\,\overline{P}_{c+1}(s)$

$c\mu\,\overline{P}_{c+1}(s) = [\ (K+Y-n)\,\lambda + c\mu + s\ )\ \ \overline{P}_c(s)\ - (K+Y-c+1)\,\lambda\,\overline{P}_{c-1}(s) - P_c(0)$

$c\,\overline{P}_{c+1}(s) = [\ (K+Y-n)\,\rho + c + \phi\ )\ \ \overline{P}_c(s)\ - [(K+Y-c+1)\,\rho + \phi]\ \overline{P}_{c-1}(s)$

where $\rho = \lambda/\mu$, $\phi = s/\mu$,

$$c\,\overline{P}_{c+1}(s) = [\ (K+Y-c)\,\rho + \phi + c]\ \frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\overline{P}_0(s)$$

$$-\ \ [(K+Y-c+1)\,\rho + \phi]\ \frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y-1}}{(Y+1)_{c-Y-1}}\overline{P}_0(s) \qquad (A)$$

$$\Rightarrow [(K+Y-c+1)\,\rho + \phi]\ \frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y-1}}{(Y+1)_{c-Y-1}}\overline{P}_0(s)$$

$$\Rightarrow\left[(K+Y-c+1)\frac{\lambda}{\mu}+\frac{s}{\mu}\right]\frac{(K\rho+\phi)^{Y}}{Y!}\frac{c(K\rho+\phi)_{c-1-Y}}{(Y+1)_{c-Y}}\overline{P}_0(s)$$

$$\Rightarrow\frac{(K\rho+\phi)^{Y}}{Y!}[(K+Y-c+1)\rho+\phi](K\rho+\phi)_{c-1-Y}\frac{c}{(Y+1)_{c-Y}}\overline{P}_0(s)$$

$$\Rightarrow\frac{(K\rho+\phi)^{Y}}{Y!}(K\rho+\phi)_{c-Y}\frac{c}{(Y+1)_{c-Y}}\overline{P}_0(s)$$

Putting in (A) we get

$$C\ \overline{P}_{c+1}(s)=[(K+Y-c)\rho+\phi+c]\frac{(K\rho+\phi)^{Y}}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\overline{P}_0(s)-$$

$$\frac{(K\rho+\phi)^{Y}}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}C\,\overline{P}_0(s)$$

$$C\ \overline{P}_{c+1}(s)=[(K+Y-c)\rho+\phi+c-c]\frac{(K\rho+\phi)^{Y}}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\overline{P}_0(s)$$

$$\overline{P}_{c+1}(s)=\frac{[(K+Y-c)\rho+\phi]}{c}\frac{(K\rho+\phi)^{Y}}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\overline{P}_0(s)$$

From equation (3.4.11)

$$P_n'(t)=-[(K+Y-n)\lambda+c\mu]P_n(t)+(N+Y-n+1)\lambda P_{n-1}(t)+c\mu P_{n+1}(t)$$

$$S\overline{P}_n(s)-P_n(0)=-[(K+Y-n)\lambda+c\mu]\overline{P}_n(s)+(K+Y-n+1)\lambda\overline{P}_{n-1}(s)+c\mu\overline{P}_{n+1}(s)$$

$$c\mu\overline{P}_{n+1}(s)=((K+Y-n)\lambda+c\mu+s)\overline{P}_n(s)-(K+Y-n+1)\lambda\overline{P}_{n-1}(s)-P_n(0)$$

$$c\mu\overline{P}_{n+1}(s)=((K+Y-n)\lambda+c\mu+s)\overline{P}_n(s)-(K+Y-n+1)\lambda\overline{P}_{n-1}(s)-s\overline{P}_{n-1}(s)$$

$$c\mu\overline{P}_{n+1}(s)=((K+Y-n)\lambda+c\mu+s)\overline{P}_n(s)-[(K+Y-n+1)\lambda+s]\overline{P}_{n-1}(s)$$

$$c\ \overline{P}_{n+1}(s) = (K+Y-n)\ \rho + c + \phi]\ \overline{P}_n(s) - [(K+Y-n+1)\ \rho + \phi\ ]\ \overline{P}_{n-1}(s)$$

$$c+1 \le n < Y+k$$

Put n = c +1

$$C\ \overline{P}_{c+2}(s) = [\ (K+Y-c-1)\ \rho + c + \phi]\ \overline{P}_{c+1}(s) - [(K+Y-c)\ \rho + \phi]\ \overline{P}_c(s)$$

$$= [\ (K+Y-c-1)\ \rho + c + \phi]\ \frac{[(K+Y-c)\rho+\phi]}{c}\ \frac{(K\rho+\phi)^Y}{Y!} *$$

$$\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\ \overline{P}_0(s) - [(K+Y-c)\ \rho+\phi]\frac{(K\rho+\phi)^Y}{Y!}\ \frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\ \overline{P}_0(s)$$

$$= [(K+Y-c)\ \rho+\phi]\frac{(K\rho+\phi)^Y}{Y!}\ \frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\ \overline{P}_0(s)\left[\frac{(K+Y-c-1)\rho+c+\phi}{c}-1\right]$$

$$= [\ (K+Y-c)\ \rho+\phi]\frac{(K\rho+\phi)^Y}{Y!}\ \frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\ \overline{P}_0(s)\ \frac{[(K+Y-c-1)\rho+\phi]}{c}$$

$$\overline{P}_{c+2}(s) = \frac{(K\rho+\phi)^Y}{Y!}\ \frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\ \frac{[(K+Y-c)\rho+\phi][(K+Y-c-1)\rho+\phi]}{c^2}\ \overline{P}_0(s)$$

Put n = c+2

$$C\ \overline{P}_{c+3}(s) = [\ (K+Y-c-2)\ \rho+c+\phi]\ \overline{P}_{c+2}(s) - [\ (K+Y-c-2+1)\ \rho+\phi]\ \overline{P}_{c+1}(s)$$

$$= [\ (K+Y-c-2)\ \rho+c+\phi]\ \frac{(K\rho+\phi)^Y}{Y!}\ \frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\ [\ (K+Y-c)\ \rho+\phi]$$

$$\frac{[(K+Y-c-1)\rho+\phi]}{c^2}\ \overline{P}_0(s)$$

$$-\ [\ (K+Y-c-1)\ \rho+\phi]\frac{(K\rho+\phi)^Y}{Y!}\ \frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\ \frac{[(K+Y-c)\rho+\phi]}{c}\ \overline{P}_0(s)$$

$$= \frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\frac{[(K+Y-c)\rho+\phi][(K+Y-c-1)\rho+\phi]}{c^2}\overline{P}_0(s)$$

$$[\,(\mathrm{K+Y\text{-}c\text{-}2})\,\rho+c+\phi-c]$$

$$\overline{P}_{c+3}(s) = \frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}[(\mathrm{K+Y\text{-}c})\,\rho+\phi][(\mathrm{K+Y\text{-}c\text{-}1})\,\rho+\phi]$$

$$\frac{[(K+Y-c-2)\rho+\phi]}{c^3}\overline{P}_0(s)$$

Similarly

$$\overline{P}_n = \overline{P}_{c+(n-c)} = \frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\frac{[(K+Y-c)\rho+\phi]_{n-c}}{c^{n-c}}\overline{P}_0(s)$$

$$\overline{P}_n = \frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\frac{[(K+Y-c)\rho+\phi]_{n-c}}{c^{n-c}}\overline{P}_0(s)$$

$$= \frac{(K\rho+\phi)^Y}{Y!}\frac{\rho^{c-Y}\left(K+\dfrac{\phi}{\rho}\right)_{c-Y}}{(Y+1)_{c-Y}}\frac{\rho^{n-c}\left[(K+Y-c)+\dfrac{\phi}{\rho}\right]_{n-c}}{c^{n-c}}$$

$$= \frac{(K\rho+\phi)^Y}{Y!}\frac{(K+\phi/\rho)_{c-Y}[(K+Y-c)+\phi/\rho]_{n-c}}{(Y+1)_{c-Y}\,c^{n-c}}\rho^{n-Y}\,\overline{P}_0(s) \quad c\le n<Y+K \qquad (3.4.15)$$

Hence, when α=0, β =1, and N=K then M/M/C/K/K with spares only then

$$\overline{P}_n = \begin{cases} \dfrac{(K\rho+\phi)^n}{n!}\,\overline{P}_0(s) & 0 \le n \le Y-1 \\[2ex] \dfrac{(K\rho+\phi)^Y}{Y!}\,\dfrac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\,\overline{P}_0(s) & Y \le n \le c-1 \\[2ex] \dfrac{(K\rho+\phi)^Y}{Y!}\,\dfrac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\,\dfrac{[(K+Y-c)\rho+\phi]_{n-c}}{c^{n-c}}\,\overline{P}_0(s) & c \le n < Y+K \end{cases}$$

(3.4.16)

$$\overline{P}_0^{-1}(s) = \sum_{n=0}^{Y-1} \frac{(K\rho+\phi)^n}{n!} + \sum_{n=Y}^{c-1} \frac{(K\rho+\phi)^Y}{Y!} \frac{\rho^{n-Y}\left(k+\frac{\phi}{\rho}\right)_{n-Y}}{(Y+1)_{n-Y}} +$$

$$\frac{(K\rho+\phi)^Y}{Y!} \frac{(K+\frac{\phi}{\rho})_{c-Y}}{(Y+1)_{c-Y}} \sum_{n=c}^{Y+K} \frac{[K+Y-c+\phi/\rho]_{n-c}}{c^{n-c}} \rho^{n-Y-c+c} \qquad \textbf{(3.4.17)}$$

$$= \sum_{n=0}^{Y-1} \frac{(K\rho+\phi)^n}{n!} + \frac{(K\rho+\phi)^Y}{Y!} \sum_{n=Y}^{c-1} \frac{\left(K+\frac{\phi}{\rho}\right)_{n-Y}}{(Y+1)_{n-Y}} \rho^{n-Y} +$$

$$\frac{(K\rho+\phi)^Y}{Y!} \frac{(K+\frac{\phi}{\rho})_{c-Y}}{(Y+1)_{c-Y}} \rho^{c-y} \sum_{n=c}^{Y+K} \frac{[K+Y-c+\phi/\rho]_{n-c}}{c^{n-c}} \rho^{n-c}$$

$$= \sum_{n=0}^{Y-1} \frac{(K\rho+\phi)^n}{n!} + \frac{(K\rho+\phi)^Y}{Y!} \sum_{n=0}^{Y-c+1} (1)_n \frac{(K+\phi/\rho)_n(\rho)^n}{(Y+1)_n n!} +$$

$$\frac{(K\rho+\phi)^Y}{Y!} \frac{(K+\frac{\phi}{\rho})_{c-Y}}{(Y+1)_{c-Y}} \rho^{c-Y} \sum_{n=0}^{c-Y-K} \frac{(K+Y-c+\phi/\rho)_n \rho^n (1)_n}{c^n n!}$$

$$=\sum_{n=0}^{Y-1} \frac{(K\rho+\phi)^n}{n!}+\frac{(K\rho+\phi)^Y}{Y!}\,{}_2F_1(1,K+\phi/\rho;\ Y+1;\ \rho\,)\ +$$

$$\frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\,{}_2F_0(1,K+Y-c+\phi/\rho;\ -\ ;\ (\rho/c)) \tag{3.4.18}$$

$$L=\lambda\overline{P}_o(s)\,\frac{\partial}{\partial\lambda}\Bigg[\sum_{n=0}^{Y-1} \frac{(K\rho+\phi)^n}{n!}+\frac{(K\rho+\phi)^Y}{Y!}\,{}_2F_1(1,K+\phi/\rho;\ Y+1;\ \rho)$$

$$+\frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\,{}_2F_0(1,K+Y-c+\phi/\rho;\ -\ ;\ \rho/c)\Bigg] \tag{3.4.19}$$

$$\text{(I)}\quad \lambda\overline{P}_o(s)\,\frac{\partial}{\partial\lambda}\sum_{n=0}^{Y-1} \frac{(K\rho+\phi)^n}{n!}=\lambda\overline{P}_o(s)\,\frac{\partial}{\partial\lambda}\sum_{n=0}^{Y-1}\frac{(K\frac{\lambda}{\mu}+\phi)^n}{n!}$$

$$=\lambda\overline{P}_o(s)\sum_{n=0}^{Y-1}\frac{n(K\frac{\lambda}{\mu}+\phi)^{n-1}K/\mu}{n!}$$

$$=\overline{P}_o(s)\sum_{n=0}^{Y-1}\frac{(K\rho+\phi)^{n-1}}{(n-1)!}K\rho$$

$$\text{(II)}\quad \lambda\overline{P}_o(s)\,\frac{\partial}{\partial\lambda}\left[\frac{(K\rho+\phi)^Y}{Y!}\,{}_2F_1(1,K+\phi/\rho;\ Y+1;\ \rho\,)\right]$$

$$=\lambda\overline{P}_o(s)\left[{}_2F_1(1,K+\phi/\rho;\ Y+1;\ \rho\,)\,\frac{\partial}{\partial\lambda}\frac{(K\rho+\phi)^Y}{Y!}+\frac{(K\rho+\phi)^Y}{Y!}\frac{\partial}{\partial\lambda}\,{}_2F_1(1,K+\phi/\rho;\ Y+1;\ \rho\,)\right]$$

$$\text{(II)a}\quad \lambda\frac{\partial}{\partial\lambda}\frac{(K\rho+\phi)^Y}{Y!}=\lambda\frac{\partial}{\partial\lambda}\frac{(K\frac{\lambda}{\mu}+\phi)^Y}{Y!}=\frac{\lambda.Y(K\frac{\lambda}{\mu}+\phi)^{Y-1}}{Y!}\frac{K}{\mu}=\frac{(K\rho+\phi)^{Y-1}}{(Y-1)!}K\rho$$

$$\text{(II)b}\quad \lambda\frac{\partial}{\partial\lambda}\,{}_2F_1(1,K+\phi/\rho;\ Y+1;\ \rho)=\frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty}\frac{(1)_n(K+\frac{\phi\mu}{\lambda})_n}{(Y+1)_n n!}\rho^n$$

$$=\lambda\sum_{n=0}^{\infty} n\frac{(1)_n(k+\phi/\rho)_n}{(y+1)_n n!}(\frac{\lambda}{\mu})^{n-1}\frac{1}{\mu}+\lambda\sum_{n=0}^{\infty}\frac{(1)_n\rho^n}{(Y+1)_n n!}(-\frac{\phi\mu}{\lambda})^n$$

$$=\sum_{n=1}^{\infty}\frac{(1)_n(K+\phi/\rho)_n}{(Y+1)_n(n-1)!}\rho^n+\lambda\sum_{n=0}^{\infty}\frac{(1)_n\rho^n}{(Y+1)_n n!}(-\frac{\phi}{\lambda})^n$$

$$\sum_{n=1}^{\infty}\frac{(1)_{n+1}(K+\phi/\rho)_{n+1}\rho^{n+1}}{(Y+1)_{n+1}n!}+\lambda\ {}_1F_1(1;Y+1;-\phi/\lambda)$$

$$=\frac{(K+\phi/\rho)\rho}{(Y+1)}\sum_{n=0}^{\infty}\frac{(2)_n(K-1+\phi/\rho)_n\rho^n}{(Y+2)_n n!}+\lambda\ {}_1F_1(1;Y+1;-\phi/\lambda)$$

$$=\frac{(K\rho+\phi)}{(Y+1)}{}_2F_1(2,K-1+\phi/\rho;Y+2;\rho)+\lambda\ {}_1F_1(1;Y+1;-\phi/\lambda)$$

Hence

$$\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\frac{(K\rho+\phi)^Y}{Y!}{}_2F_1(1,K+\phi/\rho;Y+1;\rho)=$$

$$K\rho\frac{(K\rho+\phi)^{Y-1}}{(Y-1)!}{}_2F_1(1,K+\phi/\rho;Y+1;\rho)+\frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)}{(Y+1)}{}_2F_1(2,K-1+\phi/\rho;Y+2;\rho)+$$

$$\frac{(K\rho+\phi)^Y}{Y!}\lambda\ {}_1F_1(1;Y+1;-\phi/\lambda)$$

$$=K\rho\frac{(K\rho+\phi)^{Y-1}}{(Y-1)!}{}_2F_1(1,K+\phi/\rho;Y+1;\rho)+\frac{(K\rho+\phi)^{Y+1}}{(Y+1)!}{}_2F_1(2,K-1+\phi/\rho;Y+2;\rho)+$$

$$\lambda\frac{(K\rho+\phi)^Y}{Y!}{}_1F_1(1;Y+1;-\phi/\lambda)$$

(III) $\lambda\overline{P}_0(s)\frac{\partial}{\partial\lambda}\frac{(K\rho+\phi)^Y}{Y!}\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}{}_2F_0(1,K+Y-c+\phi/\rho;-;\rho/c)$

$$=\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}{}_2F_0(1,K+Y-c+\phi/\rho;-;\rho/c)\ \lambda\frac{\partial}{\partial\lambda}\frac{(K\rho+\phi)^Y}{Y!}+$$

$$\frac{(K\rho+\phi)^{Y}}{Y!}\ \frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\ \lambda\frac{\partial}{\partial\lambda}\,{}_2F_0(1,K+Y-c+\phi/\rho;-;\rho/c)$$

(III)a $\quad \lambda\frac{\partial}{\partial\lambda}\,{}_2F_0(1,K+Y-c+\phi/\rho;-;\rho/c)$

$$\lambda\frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty}\ \frac{(1)_n(K+Y-c+\phi/\rho)_n}{n!}\left(\frac{\rho}{c}\right)^n$$

$$\lambda\sum_{n=0}^{\infty}\ \frac{(1)_n(-\phi/\rho)^n}{n!}\left(\frac{\rho}{c}\right)^n$$

$$=\lambda\sum_{n=0}^{\infty}\ \frac{(1)_n}{n!}\left(\frac{-\phi}{c}\right)^n$$

$$=\lambda\ \ {}_1F_0(1;-;-\phi/c)$$

Hence

$$L=\overline{P}_o(s)\Bigg[\sum_{n=1}^{Y-1}\ K\rho\frac{(K\rho+\phi)^{n-1}}{(n-1)!}+k\rho\frac{(K\rho+\phi)^{Y-1}}{(Y-1)!}\,{}_2F_1(1,K+\phi/\rho;Y+1;\rho)$$

$$+\ \frac{(K\rho+\phi)^{Y+1}}{(Y+1)!}\,{}_2F_1(2,K-1+\phi/\rho;Y+2;\rho)+\lambda\ \frac{(K\rho+\phi)^{Y}}{Y!}\,{}_1F_1(1;Y+1;-\phi/\lambda)+$$

$$\frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\,{}_2F_0(1,K+Y-c+\phi/\rho;-;\rho/c)\ k\rho\frac{(K\rho+\phi)^{Y-1}}{(Y-1)!}$$

$$+\ \frac{(K\rho+\phi)^{Y}}{Y!}\ \frac{(K\rho+\phi)_{c-Y}}{(Y+1)_{c-Y}}\,\lambda\ \ {}_1F_0(1;-;-\phi/c)\Bigg] \qquad (3.4.20)$$

The machine availability (rate of production per machine) is M.A. = 1 - L / K

The operative efficiency (utilization) is O.E. $= 1-\sum_{n=0}^{c-1}(1-n/c)\overline{P}_n(s)$

Chapter - 4

Transient behaviour of the M/M/C interdependent Queueing model with controllable Arrival Rates

INTRODUCTION :

Queueing models provide the basic framework for efficient design and analysis of many practical situations. Along with several other assumptions it is customary to consider that the arrival and service processes are independent. A queueing model in which arrivals and services are correlated is known as interdependent queueing model.

In interdependent queueing model, Lindley (1952) [104] analyzed a single server queueing model with recurrent input and arbitrary service times with that assumptions the service times of the n^{th} customer and time interval between arrival epoch are related by a linear function.

The present chapter discusses a queue model of the type M/M/C where, queueing system with the assumption that the arrival and service processes of the system are correlated and follow a bivariate poisson distribution. In addition to this interdependence whenever the queue size reaches a certain prescribed limit R, the arrival rate reduces from λ_0 to λ_1 and it continues with the reduced rate λ_1 as long as the number of customer in the queue is greater than some other prescribed integer r $(0 \leq r \leq R)$ and when the system reaches r, the arrival rate changes back to λ_0 and the same process is repeated. This operating strategy is called controllable arrival rates with prescribed integer R and r.

In general waiting line of a queueing model increases due to slow serving rate or fast arriving rate. There are several research papers in which the service rates are controlled to decrease the queue length. Queueing models with controllable service rate have been discussed by Gray W., Wang P Scott, M and C.M. Wang P. Many years ago, Conolly and Hadidi (1969) [38] have studied the single server queueing model with that assumption the rate S_n / T_n is constant for all n, where S_n is the n^{th} customer service time and T_n in the time interval between n^{th} and $(n+1)^{th}$ arrival epoch. Mitchel and Paulson (1979) [114] have considered and M/M/1 queueing system with an assumption that the interarrival times separating the arrival from that of his predecessor and the customer service times follows a bivariate exponential distribution. Their analysis is based on simulation of the waiting time process. Conolly and Choo (1979) [37] have provided means for exact calculation of the waiting time for a generalized correlated queueing with exponential demand and service. Lenganis (1987) [103] , Borst and Combe (1992) [25] have studied the busy period analysis of a correlated queue with exponential demand and service M.I. AftabBegum & D. Maheswari (2002) [107] have studied the M/M/C interdependent queueing model with controllable arrival rates. Here we consider the arrival and service processes of the single server infinite capacity queueing system are correlated and follows a bivariate poisson process having the joint probability mass function of the form

$$P(X_1{=}x_1, X_2{=}x_2;t) = e^{-(\lambda_i+\mu_n-e)t}$$

$$\sum_{J=0}^{\min(X_1,X_2)} (et)^J [(\lambda_i - e)t]^{(X_1-J)} \frac{[(\mu_n - e)t]^{(X_2-J)}}{J!(X_1 - J)!(X_2 - J)!}$$

with parameters λ_I (I =0,1), μ as mean arrival rate, mean service rate and mean dependence rate (covariance between the arrival and service processes) respectively. The operating strategy of the system is controllable arrival rates with prescribed integers R and r.

Transient State equation

$$P_0'(0) = -(\lambda_0 - e)\, P_0(0) + (\mu - e)\, P_1(0) \qquad (4.2.1)$$

$$P_n'(0) = -(\lambda_0 + n\mu - (n+1)e)\, P_n(0) + (\lambda_0 - e)\, P_{n-1}(0) + (n+1)(\mu - e)\, P_{n+1}(0)$$
$$1 \le n \le c-1 \qquad (4.2.2)$$

$$P_n'(0) = -(\lambda_0 + c\mu - (c+1)e)\, P_n(0) + (\lambda_0 - e)\, P_{n-1}(0) + c(\mu - e)\, P_{n+1}(0)$$
$$c \le n \le r-1 \qquad (4.2.3)$$

$$P_r'(0) = -(\lambda_0 + c\mu - (c+1)e)\, P_r(0) + (\lambda_0 - e)\, P_{r-1}(0) + c(\mu - e)\, P_{r+1}(0)$$
$$+ c(\mu - e)\, P_{r+1}(1) \qquad (4.2.4)$$

$$P_n'(0) = -(\lambda_0 + c\mu - (c+1)e)\, P_n(0) + (\lambda_0 - e)\, P_{n-1}(0) + c(\mu - e)\, P_{n+1}(0)$$
$$r+1 \le n \le R-2 \qquad (4.2.5)$$

$$P_{R-1}'(0) = -(\lambda_0 + c\mu - (c+1)e)\, P_{R-1}(0) + (\lambda_0 - e)\, P_{R-2}(0) \qquad (4.2.6)$$

$$P_{r+1}'(1) = -(\lambda_1 + c\mu - (c+1)e)\, P_{r+1}(1) + c(\mu - e)\, P_{r+2}(1) \qquad (4.2.7)$$

$$P_n'(1) = -(\lambda_1 + c\mu - (c+1)e)\, P_n(1) + c(\mu - e)\, P_{n+1}(1) + (\lambda_1 - e)\, P_{n-1}(1) \quad r+2 \le n \le R-1 \qquad (4.2.8)$$

$$P_R'(1) = -(\lambda_1 + c\mu - (c+1)e)\, P_R(1) + (\lambda_1 - e)\, P_{R-1}(1) + (\lambda_0 - e)\, P_{R-1}(0) + c(\mu - e)\, P_{R+1}(1) \qquad (4.2.9)$$

$$P_n'(1) = -(\lambda_1 + c\mu - (c+1)e)\, P_n(1) + (\lambda_1 - e)\, P_{n-1}(1) + c(\mu - e)\, P_{n+1}(1) \quad n \le R+1 \qquad (4.2.10)$$

Define boundary Condition $\quad P_n(0) = S\,\overline{P}_{n-1}(0) \qquad (4.2.11)$

From equation (4.2.1)

$$P_0'(0) = -(\lambda_0 - e)\, P_0(0) + (\mu - e)\, P_1(0)$$

Taking Laplace Transform -

$$\mathbf{S\,\overline{P}_0(0) - P(0)} = -(\lambda_0 - e)\, \overline{P}_0(0) + (\mu - e)\, \overline{P}_1(0)$$

$$(\mu-e)\,\overline{P}_1(0) = (\lambda_0-e+s)\,\overline{P}_0(0)$$

$$\overline{P}_1(0) = \left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)\overline{P}_0(0)$$

$$\overline{P}_1(0) = \left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)\overline{P}_0(0)$$

From equation (4.2.2)

$$P_n'(0) = -\,(\lambda_0+n\mu-(n+1)e)\,P_n(0) + (\lambda_0-e)\,P_{n-1}(0)$$

$$+\,(n+1)\,(\mu-e)\,P_{n+1}(0) \quad 1 \le n \le c-1$$

Taking Laplace Tarnsform -

$$S\,\overline{P}_n(0) - P_n(0) = -\,(\lambda_0+n\mu-(n+1)e)\,\overline{P}_n(0) + (\lambda_0-e)\,\overline{P}_{n-1}(0) + (n+1)\,(\mu-e)\,\overline{P}_{n+1}(0)$$

$$(\lambda_0+n\mu-(n+1)e+s)\,\overline{P}_n(0) = (\lambda_0-e)\,\overline{P}_{n-1}(0) + (n+1)\,(\mu-e)\,\overline{P}_{n+1}(0) + s\,\overline{P}_{n-1}(0)$$

$$(n+1)\,(\mu-e)\,\overline{P}_{n+1}(0) = [\lambda_0+n\mu-(n+1)e+s]\,\overline{P}_n(0) - (\lambda_0-e)\,\overline{P}_{n+1}(0) - s\,\overline{P}_{n-1}(0)$$

$$(n+1)\,\overline{P}_{n+1}(0) = [\lambda_0+n\mu-(n+1)e+s]\,/(\mu-e)\,\overline{P}_n(0) - (\lambda_0-e)/(\mu-e)\,\overline{P}_{n-1}(0) - {}^{s}\!/_{\mu-e}\,\overline{P}_{n-1}(0)$$

$$(n+1)\,\overline{P}_{n+1}(0) = [\lambda_0+n\mu-(n+1)e+s]/(\mu-e)\,\overline{P}_n(0) - (\lambda_0-e+s)/(\mu-e)\,\overline{P}_{n-1}(0)$$

put $n=1$

$$2\overline{P}_2(0) = [\lambda_0+\mu-2e+s]/(\mu-e)\overline{P}_1(0) - (\lambda_0-e+s)/(\mu-e)\overline{P}_0(0)$$

$$2\overline{P}_2(0) = \left[\frac{\lambda_0-e+\mu-e+s}{\mu-e}\right]\left[\frac{\lambda_0-e+s}{\mu-e}\right]\overline{P}_0(0) - \left(\frac{\lambda_0-e+s}{\mu-e}\right)\overline{P}_0(0) = \left[\frac{\lambda_0-e+s}{\mu-e}\right]\overline{P}_0(0)\left[\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}+1-1\right]$$

$$2\overline{P}_2(0) = \left[\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right]^2\overline{P}_0(0)$$

put $n = 2$

$$3\overline{P}_3(0) = \left[\frac{\lambda_0 + 2\mu - 3e + s}{\mu - e}\right]\overline{P}_2(0) - \left(\frac{\lambda_0 - e + s}{\mu - e}\right)\overline{P}_1(0)$$

$$3\overline{P}_3(0) = \left[\frac{\lambda_0 - e + 2(\mu - e) + s}{\mu - e}\right]\frac{1}{2}\left[\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right]^2\overline{P}_0(0) - \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)\overline{P}_0(0)$$

$$3\overline{P}_3(0) = \frac{1}{2}\left[\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right]^2\overline{P}_0(0)\left[\frac{\lambda_0 - e}{\mu - e} + 2 + \frac{s}{\mu - e} - 2\right]$$

$$\overline{P}_3(0) = \frac{1}{3!}\left[\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right]^3\overline{P}_0(0)$$

Similarly $\overline{P}_n(0) = \dfrac{1}{n!}\left[\dfrac{\lambda_0 - e}{\mu - e} + \dfrac{s}{\mu - e}\right]^n\overline{P}_0(0), \quad 0 \le n \le c-1$ (4.2.12)

From equation (4.2.3)

$$P_n'(0) = -(\lambda_0 + c\mu - (c+1)e)P_n(0) + (\lambda_0 - e)P_{n-1}(0) + c(\mu - e)P_{n+1}(0), \; c \le n \le r-1$$

Taking Laplace Transfrom

$$S\overline{P}_n(0) - P_n(0) = -(\lambda_0 + c\mu - (c+1)e)\overline{P}_n(0) + (\lambda_0 - e)\overline{P}_{n-1}(0) + c(\mu - e)\overline{P}_{n+1}(0)$$

$$c(\mu - e)\overline{P}_{n+1}(0) = (\lambda_0 + c\mu - (c+1)e + s)\overline{P}_n(0) - (\lambda_0 - e + s)\overline{P}_{n-1}(0)$$

$$c\overline{P}_{n+1}(0) = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{c(\mu - e)}{\mu - e} + \frac{s}{\mu - e}\right)\overline{P}_n(0) - \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)\overline{P}_{n-1}(0)$$

$$c\overline{P}_{n+1}(0) = \left(\frac{\lambda_0 - e}{\mu - e} + c + \frac{s}{\mu - e}\right)\overline{P}_n(0) - \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)\overline{P}_{n-1}(0)$$

put $n = c$

$$c\overline{P}_{c+1}(0) = \left(\frac{\lambda_0 - e}{\mu - e} + c + \frac{s}{\mu - e}\right)\overline{P}_c(0) - \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)\overline{P}_{c-1}(0)$$

$$c\overline{P}_{c+1}(0) = \left(\frac{\lambda_0 - e}{\mu - e} + c + \frac{s}{\mu - e}\right)\frac{1}{c!}\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^c\overline{P}_0(0)$$

$$-\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)\frac{1}{(c-1)!}\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{c-1}\overline{P}_0(0)$$

$$c\overline{P}_{c+1}(0) = \frac{1}{c!}\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^c\overline{P}_0(0)\left[\frac{\lambda_0 - e}{\mu - e} + c\frac{s}{\mu - e} - c\right]$$

$$c\overline{P}_{c+1}(0) = \frac{1}{c!}\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{c+1}\overline{P}_o(0)$$

$$\overline{P}_{c+1}(0) = \frac{1}{cc!}\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{c+1}\overline{P}_0(0)$$

put $n = c+1$

$$c\overline{P}_{c+2}(0) = \left(\frac{\lambda_0 - e}{\mu - e} + c + \frac{s}{\mu - e}\right)\overline{P}_{c+1}(0) - \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)\overline{P}_c(0)$$

$$= \left(\frac{\lambda_0 - e}{\mu - e} + c + \frac{s}{\mu - e}\right)\frac{1}{c\,c!}\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{c+1}\overline{P}_o(0) -$$

$$\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)\frac{1}{c!}\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{c}\overline{P}_o(0)$$

$$= \frac{1}{cc!}\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{c+1}\overline{P}_0(0)\left[\frac{\lambda_0 - e}{\mu - e} + c + \frac{s}{\mu - e} - c\right]$$

$$= \frac{1}{c\,c!}\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{c+2}\overline{P}_0(0)$$

$$\overline{P}_{c+2}(0) = \frac{1}{c^2 c!}\left(\frac{\lambda_0 - e}{-e} + \frac{s}{\mu - e}\right)^{c+2}\overline{P}_0(0)$$

$$Similarly\ \overline{P}_n(0) = \overline{P}_{c+(n-c)}(0) = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{n}\frac{\overline{P}_0(0)}{c^{n-c}} \quad c \le n \le r \ \mathbf{(4.2.13)}$$

From equation (4.2.4)

$$P_r'(0) = -(\lambda_0 + c\mu - (c+1)e)P_r(0) + (\lambda_0 - e)P_{r-1}(0) + c(\mu - e)\,P_{r+1}(0) + c(\mu - e)P_{r+1}(1)$$

Taking Laplace Transform

$$S\overline{P}_r(0) - P_r(0) = -(\lambda_0 + c\mu - (c+1)e)\overline{P}_r(0) + (\lambda_0 - e)\overline{P}_{r-1}(0) + c(\mu - e)\overline{P}_{r+1}(0) + c(\mu - e)\overline{P}_{r+1}(1)$$

$$c(\mu - e)\left[\overline{P}_{r+1}(0) + \overline{P}_{r+1}(1)\right] = \left[\lambda_{\_0} + c\mu - (c+1)e + s\right]\overline{P}_r(0) - (\lambda_0 - e + s)\overline{P}_{r-1}(0)$$

$$\overline{P}_{r+1}(0) + \overline{P}_{r+1}(1) = \left[\frac{\lambda_0 + c\mu - (c+1)e + s}{c(\mu - e)}\right]\overline{P}_r(0) - \left(\frac{\lambda_0 - e + s}{c(\mu - e)}\right)\overline{P}_{r-1}(0)$$

$$= \left[\frac{\lambda_0 - e + c(\mu - e) + s}{c(\mu - e)}\right]\left[\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right]^{r}\frac{P_o(0)}{c^{r-c}c!} - \left(\frac{\lambda_0 - e}{c(\mu - e)} + \frac{s}{c(\mu - e)}\right)\left[\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right]^{r-1}\frac{P_o(0)}{c^{r-1-c}\,c!}$$

$$= \left[\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right]^{r}\frac{P_0(0)}{c^{r-c}\,c!}\left[\frac{\lambda_0 - e}{c(\mu - e)} + 1 + \frac{s}{c(\mu - e)} - 1\right]$$

$$= \left[\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right]^{r+1}\frac{P_0(0)}{c^{r-c+1}c!}$$

$$\overline{P}_{r+1}(0) = \left[\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right]^{r+1}\frac{P_o(0)}{c^{r-c+1}c!} - \overline{P}_{r+1}(1)$$

From equation (4.2.5)

$$P_n'(0) = -(\lambda_0 + c\mu - (c+1)e)P_n(0) + (\lambda_0 - e)P_{n-1}(0) + c(\mu - e)P_{n+1}(0) \quad r+1 \le n \le R-2$$

Taking Laplace Transform

$$S\overline{P}_n(0) - P_n(0) = -(\lambda_0 + c\mu - (c+1)e)\overline{P}_n(0) + (\lambda_0 - e)\overline{P}_{n-1}(0) + c(\mu - e)\overline{P}_{n+1}(0)$$

$$c(\mu - e)\overline{P}_{n+1}(0) = [\lambda_0 + c\mu - (c+1)e + s]\overline{P}_n(0) - (\lambda_0 - e + s)\overline{P}_{n-1}(0)$$

put $n = r+1$

$$c(\mu-e)\overline{P}_{r+2}(0) = [\lambda_0 + c\mu - (c+1)e + s]\overline{P}_{r+1}(0) - (\lambda_0 - e + s)\overline{P}_r(0)$$

$$= [\lambda_0 + c\mu - (c+1)e + s]\left[\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{r+1} \frac{P_o(0)}{c^{r+1-c}c!} - \overline{P_{r+1}}(1)\right]$$

$$-(\lambda_0 - e + s)\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{r} \frac{P_o(0)}{c^{r-c}c!}$$

$$= \left[\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{r+1} \frac{P_0(0)}{c^{r+1-c}c!}\right][\lambda_0 + c\mu - (c+1)e + s - c(\mu - e)]$$

$$-\overline{P}_{r+1}(1)[\lambda_0 + c\mu - (c+1)e + s]$$

$$c(\mu - e)\overline{P}_{r+2}(0) = \left[\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{r+1} \frac{P_0(0)}{c^{r+1-c}c!}\right][(\lambda_0 - e + c(\mu - e) + s - c(\mu - e)]$$

$$-\overline{P}_{r+1}(1)[\lambda_0 - e + c(\mu - e) + s]$$

$$c\overline{P}_{r+2}(0) = \left[\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{r+1} \frac{P_0(0)}{c^{r+1-c}c!}\right]\left[\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\lambda - e}\right]$$

$$-\overline{P}_{r+1}(1)\left[\frac{\lambda_0 - e}{\mu - e} + c + \frac{s}{\mu - e}\right]$$

$$\overline{P}_{r+2}(0) = \left[\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{r+2} \frac{P_0(0)}{c^{r+2-c}c!} - \overline{P}_{r+1}(1)\left[\frac{\lambda_0 - e}{c(\mu - e)} + \frac{s}{c(\mu - e)} + 1\right]\right]$$

$$\overline{P}_{r+2}(0) = \left[\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{r+2} \frac{P_0(0)}{c^{r+2-c}c!} - \overline{P}_{r+1}(0)\left(1 + \frac{\lambda_0 - e}{c(\mu - e)} + \frac{s}{c(\mu - e)}\right]\right.$$

$$\overline{P}_{r+2}(0)=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{r+2}\frac{P_o(0)}{c^{r+2-c}c!}-\overline{P}_{r+1}(1)\left[\frac{1+B}{1-B}\right](1-B)$$

$$where\ B=\frac{\lambda_o-e}{c(\mu-e)}+\frac{\mathrm{s}}{\mathrm{c}(\mu\text{-}\mathrm{e})}$$

$$\overline{P}_{r+2}(0)=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{\mathrm{r+2}}\frac{P_0(0)}{c^{r+2-c}c!}-\overline{P}_{r+1}(1)\left(\frac{1-B^2}{1-B}\right)$$

Put $n=r+2$

$$c(\mu-e)\overline{P}_{r+3}(0)=\left[\lambda_0+c\mu-(c+1)e+s\right]\overline{P}_{r+2}(0)-(\lambda_0-e+s)\overline{P}_{r+1}(0)$$

$$\overline{P}_{r+3}(0)=\left[\frac{\lambda_0+c\mu-(c+1)e+s}{c(\mu-e)}\right]\left[\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{r+2}\frac{P_0(0)}{c^{r+2-c}c!}-\overline{P}_{r+1}(1)\left(\frac{1-B^2}{1-B}\right)\right]$$

$$-\left(\frac{\lambda_0-e+s}{c(\mu-e)}\right)\left[\left[\frac{\lambda o-e}{\mu-e}+\frac{s}{\mu-e}\right]^{r+1}\frac{P_0(0)}{c^{r-c+1}c!}-\overline{P}_{r+1}(1)\right]$$

$$=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{r+2}\frac{P_0(0)}{c^{r+2-c}c!}\left[\frac{\lambda_0-e+c(\mu-e)+s}{c(\mu-e}-1\right]$$

$$+\overline{P}_{r+1}(1)\left[\frac{\lambda_0-e+s}{c(\mu-e)}-\frac{\lambda_0-e+c(\mu-e)+s}{c(\mu-e)}\left(1+\frac{\lambda_0-e+s}{c(\mu-e}\right)\right]$$

$$=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{r+2}\frac{P_o(0)}{c^{r+2-c}c!}\left[\frac{\lambda_0-e+c(\mu-e)+s}{c(\mu-e)}-1\right]+$$

$$\overline{P}_{r+1}(1)\left[\frac{\lambda_0-e+s}{c(\mu-e)}-\left(1+\frac{\lambda_0-e+s}{c(\mu-e}\right)^2\right]$$

$$=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{r+2}\frac{P_0(0)}{c^{r+3-c}c!}\left[\frac{\lambda_0-e}{\mu-e}+1+\frac{s}{\mu-e}-1\right]+$$

$$\overline{P}_{r+1}(1)\left[B-(1+B)^2\right]$$

$$\overline{P}_{r+3}(0)=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{r+3}\frac{P_0(0)}{c^{r+3-c}c!}+\overline{P}_{r+1}(1)\left[B-1-B^2-2B\right]$$

$$=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{r+3}\frac{P_0(0)}{c^{r+3-c}\,c!}+\overline{P}_{r+1}(1)\left[-(1+B^2+B)\right]$$

$$\overline{P}_{r+3}(0)=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{r+3}\frac{P_0(0)}{c^{r+3-c}\,c!}-\overline{P}_{r+1}(1)\left[\frac{(1-B)(1+B^2+B)}{1-B}\right]$$

$$\overline{P}_{r+3}(0)=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{r+3}\frac{P_0(0)}{c^{r+3-c}\,c!}-\overline{P}_{r+1}(1)\left[\frac{1-B^3}{1-B}\right]$$

Similarly

$$\overline{P}_n(0)=\overline{P}_{r+n-r}(0)=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{n}\frac{\overline{P}_0(0)}{c^{n-c}c!}-\overline{P}_{r+1}(1)\left(\frac{1-B^{n-r}}{1-B}\right)$$

$$\text{where } B=\frac{\lambda_0-e}{c(\mu-e)}+\frac{s}{c(\mu-e)} \quad r+1\ \le n\le R-1 \qquad (4.2.14)$$

From equation (4.2.6)

$$P'_{R-1}(0)=-(\lambda_0+c\mu-(c+1)e)\,P_{R-1}(0)+(\lambda_0-e)P_{R-2}(0)$$

Taking Laplace Transform –

$$S\,\overline{P}_{R-1}(0)-P_{R-1}(0)=-(\lambda_0+c\mu-(c+1)e)\,\overline{P}_{R-1}(0)+(\lambda_0-e)\,\overline{P}_{R-2}(0)$$

$$-(\lambda_0+c\mu-(c+1)e+s)\,\overline{P}_{R-1}(0)+(\lambda_0-e+s)\,\overline{P}_{R-2}(0)=0$$

$$-(\lambda_0+c\mu-(c+1)e+s)\left[\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{R-1}\frac{P_o(0)}{c^{R-1-c}c!}-\overline{P}_{r+1}(1)\left(\frac{1-B^{R-1-r}}{1-B}\right)\right]$$

$$+(\lambda_0-e+s)\left[\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{R-2}\frac{P_0(0)}{c^{R-2-c}\,c!}-\overline{P}_{r+1}(1)\left(\frac{1-B^{R-2-r}}{1-B}\right)\right]=0$$

$$-(\lambda_0+c\mu-(c+1)e+s)\left[\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{R-1}\frac{P_0(0)}{c^{R-1-c}c!}\right]+(\lambda_0-e+s)\left[\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^{R-2}\frac{P_o(0)}{c^{R-2-c}c!}\right]+$$

$$\overline{P}_{r+1}(1)\left[(\lambda_0 + c\mu - (c+1)e + s)\left(\frac{1-B^{R-1-r}}{1-B}\right) - (\lambda_0 - e + s)\left(\frac{1-B^{R-2-r}}{1-B}\right)\right] = 0$$

$$\left[\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R-1} \frac{P_0(0)}{c^{R-1-c}c!}[-\lambda_0 - c(\mu - e) + e - s + c(\mu - e)\right]$$

$$+\overline{P}_{r+1}(1)\left[\{(\lambda_0 - e) + c(\mu - e) + s\}\left(\frac{1-B^{R-1-r}}{1-B}\right) - (\lambda_0 - e + s)\left(\frac{1-B^{R-2-r}}{1-B}\right)\right] = 0$$

$$-\left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R-1} \frac{P_o(0)}{c^{R-1-c}c!} + \overline{P}_{r+1}(1)\left[\left(1 + \frac{c(\mu - e}{\lambda_0 - e + s}\right)\left(\frac{1-B^{R-1-r}}{1-B}\right) - \left(\frac{1-B^{R-2-r}}{1-B}\right)\right] = 0$$

$$\overline{P}_{r+1}(1)\left[\left(1 + \frac{1}{B}\right)\left(\frac{1-B^{R-1-r}}{1-B}\right) - \frac{(1-B^{R-2-r})}{1-B}\right] = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R-1} \frac{P_o(0)}{c^{R-1-c}\,c!}$$

$$\overline{P}_{r+1}(1)\left[\left(\frac{1+B}{B}\right)\left(\frac{1-B^{R-1-r}}{1-B}\right) - \left(\frac{1-B^{R-2-r}}{1-B}\right)\right] = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R-1} \frac{P_o(0)}{c^{R-1-c}c!}$$

$$\overline{P}_{r+1}(1)\left[\frac{(1+B)(1-B^{R-1-r}) - B(1-B^{R-2-r})}{B(1-B)}\right] = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R-1} \frac{P_o(0)}{c^{R-1-c}c!}$$

$$\overline{P}_{r+1}(1)\left[\frac{1-B^{R-1-r} + B - B^{R-r} - B + B^{R-1-r}}{B\,(1\text{-}\mathrm{B})}\right] = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R-1} \frac{P_0(0)}{c^{R-1-c}c!}$$

$$\overline{P}_{r+1}(1)\left[\frac{1-B^{R-r}}{B(1-B)}\right] = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R-1} \frac{P_o(0)}{c^{R-1-c}c!}$$

$$\overline{P}_{r+1}(1) = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R-1} \frac{P_0(0)}{c^{R-1-c}c!}\frac{B(1-B)}{(1-B^{R-r})}$$

$$\overline{P}_{r+1}(1) = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R} \frac{P_0(0)}{c^{R-c}c!}\frac{c(\mu - e)}{(\lambda_0 - e + s)}\frac{B(1-B)}{(1-B^{R-r})}$$

$$\overline{P}_{r+1}(1) = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R} \frac{P_0(0)}{c^{R-c}c!}\frac{1}{B}\frac{B(1-B)}{(1-B^{R-r})}$$

$$\overline{P}_{r+1}(1) = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R} \frac{P_0(0)}{c^{R-c}c!}\frac{B^r(1-B)}{B^r(1-B^{R-r})}$$

$$\overline{P}_{r+1}(1) = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R} \frac{P_0(0)}{c^{R-c}c!}\frac{B^r(1-B)}{(B^r - B^R)}$$

$$\overline{P}_{r+1}(1) = \left(\frac{\lambda_0 - e}{\mu - e} + \frac{s}{\mu - e}\right)^{R} \frac{P_0(0)}{c^{R-c}c!}B^r(1-B)\left[B^r - B^R\right]^{-1}$$

$$\overline{P}_{r+1}(1)=\left(\frac{\lambda_0-e}{c(\mu-e)}+\frac{s}{c(\mu-e)}\right)^R\frac{P_0(0)}{c!}c^c\,B^r\,(1-B)\left[B^r-B^R\right]^{-1}$$

$$=B^R\,\frac{P_0(0)}{c!}c^c\,B^r\,(1-B)\left[B^r-B^R\right]^{-1}$$

$$=B^{R+r}\,(1-B)\frac{c^c}{c!}\left[B^r-B^R\right]^{-1}\,P_0(0)$$

$$\overline{P}_{r+1}\,(1)=B^{R+r}\,(1-B)\frac{c^c}{c!}A\,P_0\,(0) \qquad (4.2.15)$$

$$where\ A=\left[B^r-B^R\right]^{-1}$$

Substituting Equation (4.2.12) in Equation (4.2.11) then we get.

$$P_n(0)=\left(\frac{\lambda_0-e}{\mu-e}+\frac{s}{\mu-e}\right)^n\frac{P_0(0)}{c^{n-c}c!}-B^{R+r}(1-B)\frac{c^c}{c!}AP_o(0)\left(\frac{1-B^{n-r}}{1-B}\right)$$

$$P_n(0)=\left(\frac{\lambda_0-e}{c(\mu-e)}+\frac{s}{c(\mu-e)}\right)^nP_o(0)\frac{c^c}{c!}-B^{R+r}(1-B)\frac{c^c}{c!}AP_0(0)\left(\frac{1-B^{n-r}}{1-B}\right)$$

$$P_n(0)=B^nP_0(0)\frac{c^c}{c!}-B^{R+r}(1-B)\frac{c^c}{c!}AP_0(0)\left[\frac{1-B^{n-r}}{1-B}\right]$$

$$=B^nP_0(0)\frac{c^c}{c!}-B^{R+r}\frac{c^c}{c!}(B^r-B^R)^{-1}P_0(0)(1-B^{n-r})$$

$$=B^rB^{n-r}P_0(0)\frac{c^c}{c!}-B^{R+1}\frac{c^c}{c!}(B^r-B^R)^{-1}P_0(0)(1-B^{n-r})$$

$$=B^rB^{n-r}P_0(0)\frac{c^c}{c!}\frac{(B^r-B^R)^{-1}}{(B^r-B^R)^{-1}}-B^{R+r}\frac{c^c}{c!}(B^r-B^R)^{-1}P_0(0)(1-B^{n-r})$$

$$=A\,P_0(0)\frac{c^c}{c!}\left[\frac{B^rB^{n-r}}{(B^r-B^R)^{-1}}-B^{R+r}(1-B^{n-r})\right]$$

$$=AP_o(0)\frac{c^c}{c!}\left[\frac{B^n}{(B^r-B^R)^{-1}}-B^{R+r}(1-B^{n-r})\right]$$

$$= AP_0(0)\frac{c^c}{c!}\left[B^n(B^r - B^R) - B^{R+r}(1-B^{n-r})\right]$$

$$= AP_0(0)\frac{c^c}{c!}\left[B^{n+r} - B^{n+R} - B^{R+r} + B^{n+R}\right]$$

$$= AP_0(0)\frac{c^c}{c!}B^r(B^n - B^R)$$

Similarly

$$P_n(0) = B^r(B^n - B^R)\frac{c^c}{c!}AP_o(0) \quad r+1 \le n \le R-1$$

$$\text{where } A = \left[B^r - B^R\right]^{-1} \qquad (4.2.16)$$

$$B = \frac{\lambda_0 - e}{c(\mu - e)} + \frac{s}{c(\mu - e)}$$

From equation (4.2.7)

$$P'_{r+1}(1) = (\lambda_1 + c\mu - (c+1)e)P_{r+1}(1) + c(\mu - e)p_{r+2}(1)$$

Taking Laplace *Transform* –

$$S\overline{P}_{r+1}(1) - P_{r+1}(0) = -(\lambda_1 + c\mu - (c+1)e)\overline{P}_{r+1}(1) + c(\mu - e)\overline{P}_{r+2}(1)$$

$$c(\mu - e)\overline{p}_{r+2}(1) = (\lambda_1 + c\mu - (c+1)e + s)\overline{P}_{r+1}(1)$$

$$\overline{P}_{r+2}(1) = \left(\frac{\lambda_1 - e + c(\mu - e) + s}{c(\mu - e)}\right)\overline{P}_{r+1}(1)$$

$$\overline{P}_{r+2}(1) = (1 + D + D')\overline{P}_{r+1}(1)$$

$$\text{where } D = \frac{\lambda_1 - e}{c(\mu - e)} \quad D' = \frac{s}{c(\mu - e)}$$

From equation (4.2.8)

$$P'_n(1) = -(\lambda_1 + c\mu - (c+1)e)P_n(1) + c(\mu - e)P_{n+1}(1) + (\lambda_1 - e)P_{n-1}(1)$$
$$r+2 \le n \le R-1$$

Taking Laplace Transform –

$$S\overline{P}_n(1) - P_n(0) = -[\lambda_1 + c\mu - (c+1)e)\overline{P}_n(1) + c(\mu - e)\overline{P}_{n+1}(1) +$$
$$(\lambda_1 - e)\overline{P}_{n-1}(1)$$

$$S\overline{P}_n(1) - S\overline{P}_{n-1}(1) = -(\lambda_1 + c\mu - (c+1)e)\overline{P}_n(1) + c(\mu - e)\overline{P}_{n+1}(1) +$$
$$(\lambda_1 - e)\overline{P}_{n-1}(1)$$

$$\overline{P}_{n+1}(1)=\left(\frac{\lambda_1-e+c(\mu-e)+s}{c(\mu-e)}\right)\overline{P}_n(1)-\left(\frac{\lambda_1-e+s}{c(\mu-e)}\right)\overline{P}_{n-1}(1)$$

$$\overline{P}_{n+1}(1)=(1+D+D')\overline{P}_n(1)-(D+D')\overline{P}_{n-1}(1)$$

Put $n=r+2$

$$\overline{P}_{r+3}(1)=(1+D+D')\overline{P}_{r+2}(1)-(D+D')\overline{P}_{r+1}(1)=(1+D+D')(1+D+D')\overline{P}_{r+1}(1)-(D+D')\overline{P}_{r+1}(1)$$

$$\overline{P}_{r+3}(1)=\left[(1+D+D')^2-(D+D')\right]\overline{P}_{r+1}(1)$$

$$\overline{P}_{r+3}(1)=\left[1+D^2+D'^2+2D+2DD'+2D'-D-D'\right]\overline{P}_{r+1}(1)$$

$$\overline{P}_{r+3}(1)=\left[1+D^2+D'^2+2DD'+D+D'\right]\overline{P}_{r+1}(1)$$

$$=\left[(1+D+D')+(D+D')^2\right]\overline{P}_{r+1}(1)$$

$$\overline{P}_{r+3}(1)=\left(1-(D+D')\right)\left[\frac{(1+(D+D\ )^2+(D+D'))}{(1-(D+D^1))}\right]\overline{P}_{r+1}(1)$$

$$\overline{P}_{r+3}(1)=\frac{1-(D+D')^3}{1-(D+D')}\overline{P}_{r+1}(1)$$

Put $n=r+3$

$$\overline{P}_{r+4}(1)=(1+D+D')\overline{P}_{r+3}(1)-(D+D')\overline{P}_{r+2}(1)$$

$$=(1+D+D')\left[(1+D+D')^2-(D+D')\right]\overline{P}_{r+1}(1)-(D+D')(1+D+D')\overline{P}_{r+1}(1)$$

$$=\left[(1+D+D')^3-(D+D')(1+D+D')-(D+D')(1+D+D')\right]\overline{P}_{r+1}(1)$$

$$=\left[(1+D+D')^3-2(1+D+D')(D+D')\right]\overline{P}_{r+1}(1)$$

$$=(1+D+D')\left[(1+D+D')^2-2(D+D')\right]\overline{P}_{r+1}(1)$$

$$=\frac{[1-(D+D')](1+D+D')[(1+D+D')^2-2(D+D')]}{[1-(D+D')]}\overline{P}_{r+1}(1)$$

$$=\frac{[1-(D+D')^2][1+(D+D')^2]}{[1-(D+D')]}\overline{P}_{r+1}(1)$$

$$\overline{P}_{r+4}(1)=\frac{1-(D+D')^4}{1-(D+D')}\overline{P}_{r+1}(1)$$

Similarly

$$\overline{P}_n(1)=\overline{P}_{r+(n-r)}(1)=\overline{P}_{r+1}(1)\left[\frac{1-(D+D')^{n-r}}{1-(D+D')}\right]\qquad r+1\le n\le R \qquad (4.2.17)$$

Where $D=\dfrac{\lambda_1-e}{c(\mu-e)}$ $D'=\dfrac{s}{c(\mu-e)}$

From equation (4.2.9)

$$P'_R(1) = -(\lambda_1 + c\mu - (c+1)e)P_R(1) + (\lambda_1 - e)P_{R-1}(1) + (\lambda_0 - e)P_{R-1}(0) + c(\mu - e)P_{R+1}(1)$$

Taking Laplace Transform

$$S\overline{P}_R(1) - P_R(0) = -(\lambda_1 - e + c(\mu - e)\overline{P}_R(1) + (\lambda_1 - e)\overline{P}_{R-1}(1) + (\lambda_0 - e)\overline{P}_{R-1}(0) + c(\mu - e)R_{R+1}(1)$$

$$S\overline{P}_R(1) - S\overline{P}_{R-1}(1) = -(\lambda_1 - e + c(\mu - e))\overline{P}_R(1) + (\lambda_1 - e)\overline{P}_{R-1}(1) + (\lambda_0 - e)\overline{P}_{R-1}(0) + c(\mu - e)\overline{P}_{R+1}(1)$$

$$c(\mu - e)\overline{P}_{R+1}(1) = [\lambda_1 - e + c(\mu - e) + s]\overline{P}_R(1) - (\lambda_1 - e + s)\overline{P}_{R-1}(1) - (\lambda_0 - e)\overline{P}_{R-1}(0)$$

$$\overline{P}_{R+1}(1) = (1 + D + D')\overline{P}_R(1) - (D + D')\overline{P}_{R-1}(1) - B\overline{P}_{R-1}(0)$$

From equation (4.2.10)

$$P'_n(1) = -(\lambda_1 + c\mu - (c+1)e)P_n(1) + (\lambda_1 - e)P_{n-1}(1) + c(\mu - e)P_{n+1}(1) \quad n \geq R+1$$

Taking Laplace Transform –

$$S\overline{P}_n(1) - P_n(0) = -(\lambda_1 + c\mu - (c+1)e)\overline{P}_n(1) + (\lambda_1 - e)\overline{P}_{n-1}(1) + c(\mu - e)\overline{P}_{n+1}(1)$$

$$c(\mu - e)\overline{P}_{n+1}(1) = (\lambda_1 - e + c(\mu - e) + s)\overline{P}_n(1) - (\lambda_1 - e + s)\overline{P}_{n-1}(1)$$

$$\overline{P}_{n+1}(1) = (1 + D + D')\overline{P}_n(1) - (D + D'))\overline{P}_{n-1}(1)$$

Put $n = R + 1$

$$\overline{P}_{R+2}(1) = (1 + D + D')\overline{P}_{R+1}(1) - (D + D')\overline{P}_R(1)$$

$$= (1 + D + D')\left[(1 + D + D')\overline{P}_R(1) - (D + D')\overline{P}_{R-1}(1) - B\overline{P}_{R-1}(0)\right] - (D + D')\overline{P}_R(1)$$

Put $D + D' = K$

$$\overline{P}_{R+2}(1) = (1+K)[(1+K)\overline{P}_R(1) - K\overline{P}_{R-1}(1) - B\overline{P}_{R-1}(0)] - K\overline{P}_R(1)$$

$$\overline{P}_{R+2}(1) = \left[(1+K)^2 - K\right]\overline{P}_R(1) - K(1+K)\overline{P}_{R-1}(1) - B(1+K)\overline{P}_{R-1}(0)$$

$$= \left[1 + K + K^2\right]\overline{P}_R(1) - K(1+K)\overline{P}_{R-1}(1) - B(1+K)\overline{P}_{R-1}(0)$$

$$= (1 + K + K^2)\overline{P}_{r+1}(1)\left[\frac{1 - K^{R-r}}{1 - K}\right] - K(1+K)\overline{P}_{r+1}(1)\left(\frac{1 - K^{R-1-r}}{1 - K}\right)$$

$$- B(1+K)B'(B^{R-1} - B^R)\frac{c^c}{c!}AP_o(0)$$

$$= (1+K+K^2)\overline{P}_{r+1}(1)\left[\frac{1-K^{R-r}}{1-K}\right] - K(1+K)\overline{P}_{r+1}(1)\left(\frac{1-K^{R-1-r}}{1-K}\right)$$

$$-(1+K)B^{r+1}B^{R-1}(1-B)\frac{c^c}{c!}AP_0(0)$$

$$= (1+K+K^2)\overline{P}_{r+1}(1)\left[\frac{1-K^{R-r}}{1-K}\right] - K(1+K)\overline{P}_{r+1}(1)\left(\frac{1-K^{R-1-r}}{1-K}\right) - (1+K)\overline{P}_{r+1}(1)\frac{(1-K)}{1-K}$$

$$= \frac{\overline{P}_{r+1}(1)}{1-K}\left[(1+K+K^2)(1-K^{R-r}) - K(1+K)(1-K^{R-1-r}) - (1-K^2)\right]$$

$$= \frac{\overline{P}_{r+1}(1)}{1-K}\left[1-K^{R-r}+K-K^{R-r+1}+K^2-K^{R-r+2}-K+K^{R-r}-K^2+K^{R-r+1}-1+K^2\right]$$

$$\overline{P}_{R+2}(1) = \overline{P}_{r+1}(1)\left[\frac{K^2-K^{R+2-r}}{1-K}\right]$$

$$\overline{P}_{R+2}(1) = \overline{P}_{r+1}(1)\left[\frac{K^{R+2-R}-K^{R+2-r}}{1-K}\right]$$

Put n=R+2

$$\overline{P}_{R+3}(1)=(1+K)\overline{P}_{R+2}(1)-K\overline{P}_{R+1}(1)$$

$$=(1+K)\left[\frac{K^2-K^{R+2-r}}{1-K}\right]\overline{P}_{r+1}(1)-K\left[(1+K)\overline{P}_R(1)-K\overline{P}_{R-1}(1)-B\overline{P}_{R-1}(0)\right]$$

$$=(1+K)\left[\frac{K^2-K^{R+2-r}}{1-K}\right]\overline{P}_{r+1}(1)-K\left[(1+K)\overline{P}_{r+1}(1)\left(\frac{1-K^{R-r}}{1-K}\right)-K\overline{P}_{r+1}(1)\left(\frac{1-K^{R-1-r}}{1-K}\right)-B\overline{P}_{R-1}(0)\right]$$

$$=(1+K)\left[\frac{K^2-K^{R+2-r}}{1-K}\right]\overline{P}_{r+1}(1)-K(1+K)\overline{P}_{r+1}(1)\left(\frac{1+K^{R-r}}{1-K}\right)+K^2\overline{P}_{r+1}(1)\left(\frac{1-K^{R-1-r}}{1-K}\right)+BKB^r(B^{R-1}-B^R)\frac{c^c}{c!}AP_0(0)$$

$$=(1+K)\left[\frac{K^2-K^{R+2-r}}{1-K}\right]\overline{P}_{r+1}(1)-K(1+K)\overline{P}_{r+1}(1)\left(\frac{1-K^{R-r}}{1-K}\right)+K^2\overline{P}_{r+1}(1)\left(\frac{1-K^{R-1-r}}{1-K}\right)+KB^{r+R}(1-B)\frac{C^c}{C!}AP_0(0)$$

$$=(1+K)\left[\frac{K^2-K^{R+2-r}}{1-K}\right]\overline{P}_{r+1}(1)-K(1+K)\overline{P}_{r+1}(1)\left(\frac{1-K^{R-r}}{1-K}\right)+K^2\overline{P}_{r+1}(1)\left(\frac{1-K^{R-1-r}}{1-K}\right)+K\overline{P}_{r+1}(1)\left(\frac{1-K}{1-K}\right)$$

$$=\frac{\overline{P}_{r+1}(1)}{1-K}\left[(1+K)(K^2-K^{R+2-r})-K(1+K)(1-K^{R-r})+K^2\left(\frac{1-K^{R-1-r}}{1-K}\right)+K(1-K)\right]$$

$$=\frac{\overline{P}_{r+1}(1)}{1-K}\left[K^2-K^{R+2-r}+K^3-K^{R+3-r}+K^{R-r+1}-K^2+K^{R-r+2}+K^2-K^{R-r+1}+K-K^2-K\right]$$

$$=\overline{P}_{r+1}(1)\left[\frac{K^3-K^{R+3-r}}{1-K}\right]$$

$$\overline{P}_{R+3}(1) = \overline{P}_{r+1}(1)\left[\frac{K^3 - K^{R+3-r}}{1-K}\right]$$

$$\overline{P}_{R+3}(1) = \overline{P}_{r+1}(1)\left[\frac{K^{R+3-R} - K^{R+3-r}}{1-K}\right]$$

Similarly

$$\overline{P}_n(1) = \overline{P}_{R+(n-R)}(1) = \overline{P}_{r+1}(1)\left[\frac{K^{n-R} - K^{n-r}}{1-K}\right]$$

$$= \overline{P}_{r+1}(1)\left[K^n\left(\frac{K^{-R} - K^{-r}}{1-K}\right)\right]$$

$$\overline{P}_n(1) = \overline{P}_{r+1}(1)\left[(D+D^1)^n\left[\frac{(D+D^1)^{-R} - (D+D^1)^{-r}}{1-(D+D^1)}\right]\right] \qquad n \geq R \qquad (4.2.18)$$

Performance measures of the Model : -

Using *the system size distribution we can analyze the system behaviour of the* mod*el by deriving the system characteristics under equilibrium.* 1)*The probability that the system is in state* $(i = 0,1)$ is

$$P(i) = \sum_{n=0}^{\infty} \overline{p}_n(i), (i = 0,1)$$

Therefore

$$\overline{P}(0) = \sum_{n=0}^{c-1} \overline{P}_n(0) + \sum_{n=c}^{r} \overline{P}_n(0) + \sum_{n=r+1}^{R-1} \overline{P}_n(0)$$

$$\overline{P}(0) = \sum_{n=0}^{c-1} \frac{(Bc)^n}{n!} \overline{P}_0(0) + \frac{c^c}{c!} \overline{P}_0(0) \sum_{n=c}^{r} B^n + \frac{c^c}{C!} A\overline{P}_0(0) B^r \sum_{n=r+1}^{R-1} (B^n - B^R)$$

$$\sum_{n=c}^{r} B^n = B^c + B^{c+1} + B^{c+2} + ---- + B^r = \frac{B^c - B^{r+1}}{1-B}$$

$$\sum_{n=r+1}^{R-1} (B^n - B^R) = (B^{r+1} - B^R) + (B^{r+2} - B^R) + ---- + (B^{R-1} - B^R) = \frac{B^{r+1} - B^{R-1}B}{1-B} - B^R(R-r)$$

$$\overline{P}(0) = \sum_{n=0}^{c-1} \frac{(Bc)^n}{n!} \overline{P}_0(0) + \frac{c^c}{C!} \overline{P}_0(0)\left[\frac{B^c - B^{r+1}}{1-B}\right] + \frac{c^c}{c!} A\overline{P}_0(0) B^r\left[\frac{B^{r+1} - B^{R+1}}{1-B} - B^R(R-r)\right]$$

$$= \sum_{n=0}^{c-1} \frac{(BC)^n}{n!} \overline{P}_0(0) + \left(\frac{c^c}{c!}\right) \overline{P}_o(0) \left[\frac{B^c - B^{r+1}}{1-B}\right] +$$

$$\frac{c^c}{c!}\left[B^r - B^R\right]^{-1} \overline{P}_0(0) B^{r+1} \left[\frac{B^r - B^R}{1-B}\right]$$

$$-\frac{c^c}{c!}\left[B^r - B^R\right]^{-1} \overline{P}_0(0) B^{R+r}(R-r)$$

$$= \sum_{n=0}^{c-1} \frac{(BC)^n}{n!} \overline{P}_0(0) + \left(\frac{c^c}{c!}\right) \overline{P}_0(0) \frac{B^c}{1-B} - \left(\frac{c^c}{c!}\right) \overline{P}_0(0) \frac{B^{r+1}}{1-B} +$$

$$\frac{c^c}{c!} \overline{P}_0(0) \frac{B^{r+1}}{1-B} - \frac{c^c}{c!}(B^r - B^R)^{-1} \overline{P}_0(0) B^{R+r}(R-r)$$

$$\overline{P}(0) = \sum_{n=0}^{c-1} \frac{(BC)^n}{n!} \overline{P}_0(0) + \frac{C^c}{C!}\left[\frac{B^c}{1-B} - A(R-r)B^{R+r}\right] \overline{P}_0(0) \qquad (4.2.19)$$

Also

$$\overline{P}(1) = \sum_{n=r+1}^{R} \overline{P}_n(1) + \sum_{n=R+1}^{\infty} \overline{P}_n(1)$$

$$= \sum_{n=r+1}^{R} \overline{P}_{r+1}(1) \left[\frac{1-(D+D')^{n-r}}{1-(D+D')}\right] +$$

$$\sum_{n=R+1}^{\infty} \overline{P}_{r+1}(1) \left[(D+D')^n \left[\frac{(D+D')^{-R} - (D+D')^{-r}}{1-(D+D')}\right]\right]$$

Let $D + D' = \Phi$

$$\overline{P}(1) = \sum_{n=r+1}^{R} \overline{P}_{r+1}(1) \left[\frac{1-\phi^{n-r}}{1-\phi}\right] + \sum_{n=R+1}^{\infty} \overline{P}_{r+1}(1) \Phi^n \left[\frac{\Phi^{-R} - \Phi^{-r}}{1-\Phi}\right]$$

$$= \overline{P}_{r+1}(1) \left[\sum_{n=r+1}^{R} \frac{1-\Phi^{n-r}}{1-\Phi} + \sum_{n=R+1}^{\infty} \Phi^n \left(\frac{\Phi^{-R} - \Phi^{-r}}{1-\Phi}\right)\right]$$

$$= \overline{P}_{r+1}(1) \left[\frac{1}{(1-\Phi)} \left[(1-\Phi) + (1-\Phi^2) + --- + (1-\Phi^{R-r})\right] +\right.$$

$$\left.\left(\frac{\Phi^{-R} - \Phi^{-r}}{1-\Phi}\right) \left[\Phi^{R+1} + \Phi^{R+2} + --- \infty\right]\right]$$

$$= \overline{P}_{r+1}(1) \left[\frac{1}{(1-\Phi}\left[(1+1+---+1) - (\Phi + \Phi^2 + --- + \Phi^{R-r})\right] +\right.$$

$$\left(\frac{\Phi^{-R}-\Phi^{-r}}{1-\Phi}\right)\Phi^{R+1}(1+\Phi+\Phi^2+---\infty)\ ]$$

$$=\overline{P}_{r+1}(1)\left[\frac{1}{(1-\Phi)}\left[(R-r)-\left(\frac{\Phi-\Phi^{R-r+1}}{1-\Phi}\right)\right]+\right.$$

$$\left.\left(\frac{\Phi^{-R}-\Phi^{-r}}{1-\Phi}\right)\Phi^{R+1}\left(\frac{1}{1-\Phi}\right)\right.$$

$$=\overline{P}_{r+1}(1)\left[\frac{(R-r)(1-\Phi)-\Phi+\Phi^{R-r+1}+\Phi-\Phi^{R-r+1}}{(1-\Phi)^2}\right]$$

$$\overline{P}(1)=\overline{P}_{r+1}(1)\left(\frac{R-r}{1-\Phi}\right)=\overline{P}_{r+1}(1)\frac{R-r}{1-(D+D')} \qquad (4.2.20)$$

(ii)The probabiliy $[\overline{P}_o(0)]$ that the systemis empty can be calculated from the normalizing condition –

$$\sum_{n=0}^{R-1}\overline{P}_n(0)+\sum_{n=r+1}^{\infty}\overline{P}_n(1)=\overline{P}(0)+\overline{P}(1)=1$$

Substituting for $\overline{P}(0)$ and $\overline{P}(1)$ in the above equation and simplifying

$$=\sum_{n=0}^{c-1}\frac{(BC)^n}{n!}\overline{P}_0(0)+\frac{c^c}{c!}\left(\frac{B^c}{1-B}-A(R-r)B^{R+r}\right)\overline{P}_0(0)$$
$$+\overline{P}_{r+1}(1)((R-r)/(1-(D+D')))$$

$$=\sum_{n=0}^{c-1}\frac{(BC)^n}{c!}\overline{P}_0(0)+\frac{c^c}{c!}\left(\frac{B^c}{1-B}-A(R-r)B^{R+r}\right)\overline{P}_0(0)+$$

$$B^{R+r}(1-B)\left(\frac{c^c}{c!}\right)A\overline{P}_0(0)\,((R-r)/(1-(D+D')))$$

$$=\sum_{n=0}^{c-1}\frac{(BC)^n}{n!}\overline{P}_0(0)+\frac{c^c}{c!}\left(\frac{B^c}{1-B}-A(R-r)B^{R+r}\right)\overline{P}_0(0)+$$

$$\left(\frac{\lambda_0-e}{c(\mu-e)}+\frac{s}{c(\mu-e)}\right)^R\frac{c^c}{c!}\frac{B^r(1-B)}{(B^r-B^R)}\frac{R-r}{(1-(D+D'))}\overline{P}_0(0)$$

$$\left[\overline{P}_0(0)\right]^{-1} = \sum_{n=0}^{c-1}\frac{(BC)^n}{n!} + \frac{C^C}{c!}\left\{\frac{B^c}{1-B} - A(R-r)B^{R+r}\right\} +$$

$$B^R \frac{C^c}{c!}\frac{B^r(1-B)}{(B^r-B^R)}\frac{R-r}{1-(D+D')}$$

$$= \sum_{n=0}^{c-1}\frac{(BC)^n}{n!} + \frac{C^c}{c!}\left\{\frac{B^c}{1-B} - A(R-r)B^{R+r}\left(1-\frac{1-B}{1-(D+D')}\right)\right\}$$

$$= \sum_{n=0}^{c-1}\frac{(BC)^n}{n!} + \frac{C^C}{c!}\left\{\frac{B^c}{1-B} - A(R-r)B^{R+r}\left(\frac{1-(D+D')-1+B}{1-(D+D')}\right)\right.$$

$$= \sum_{n=0}^{c-1}\frac{(BC)^n}{n!} + \frac{c^c}{c!}\left\{\frac{B^c}{1-B} - A(R-r)B^{R+r}\left(\frac{B-(D+D')}{1-(D+D')}\right)\right\} \qquad (4.2.21)$$

(*iii*)**The probability that the numberof units in the systemlies betweenr and R is obtainedas**

$$\overline{P}(r \le n \le R) = \sum_{n=r}^{R-1}\overline{P}_n(0) + \sum_{n=r+1}^{R}\overline{P}_n(1)$$

$$= \sum_{n=r}^{R-1}B^r(B^n - B^R)\frac{c^c}{c!}A\overline{P}_0(0) + \sum_{n=r+1}^{R}\overline{P}_{r+1}(1)\left[\frac{1-(D+D')^{n-r}}{1-(D+D')}\right]$$

$$= B^r\frac{c^c}{c!}A\overline{P}_0(\text{o})\sum_{n=r}^{R-1}(B^n - B^R) + \frac{\overline{P}_{r+1}(1)}{1-(D+D')}\sum_{n=r+1}^{R}\left[1-(D+D')^{n-r}\right]$$

$$= B^r\frac{c^c}{c!}A\overline{P}_0(0)\left[(B^r - B^R) + ---- + (B^{R-1} - B^R)\right] + \frac{\overline{P}_{r+1}(1)}{1-(D+D')}$$

$$\left[\{(1-(D+D')\} + \{(1-(D+D')^2\} + \{1-(D+D')^3\} + ---- + \{1-(D+D')^{R-r}\}\right]$$

$$= B^r\frac{c^c}{c!}A\overline{P}_0(0)\left[(B^r + B^{r+1} + ---- + B^{R-1}) - (B^R + B^R + ---- + B^R)\right]$$

$$+ \frac{\overline{P}_{r+1}(1)}{1-(D+D')}\left[R-r-1\sum_{n=r+1}^{R}(D+D')^{n-r}\right]$$

$$= B^r\frac{c^c}{c!}A\overline{P}_0(0)\left[\left(\frac{B^r - B^{R-1}B}{1-B}\right) - (R-1-r)B^R\right] +$$

$$\frac{\overline{P}_{r+1}(1)}{1-(D+D')}\left[R-r-1-\sum_{n=r+1}^{R}(D+D')^{n-r}\right]$$

$$= B^r\frac{c^c}{c!}A\overline{P}_0(0)\left[\left(\frac{B^r - B^R}{1-B}\right) - (R-1-r)B^R\right] + \frac{\overline{P}_{r+1}(1)}{1-(D+D')}\left[R-r-1-\sum_{n=1}^{R-r}(D+D')^n\right]$$

$$= B^r\frac{c^c}{c!}A\overline{P}_0(0)\left[\left(\frac{B^r - B^R}{1-B}\right) - (R-1-r)B^R\right] + \frac{AB^{R+r}(1-B)\frac{c^c}{c!}\overline{P}_0(0)}{1-(D+D')}\left[R-r-1-\sum_{n=1}^{R-r}(D+D')^n\right]$$

$$= \overline{P}_o(0)\left(\frac{c^c}{c!}B^r\right)\left[A\frac{(B^r - B^R)}{1-B} - A(R-1-r)B^R + \frac{AB^R(1-B)}{1-(D+D')}\left[R-r-1-\sum_{n=1}^{R-r}(D+D')^n\right]\right]$$

$$= \overline{P}_0(0)\left(\frac{C^c}{C!}\right)B^r\left[\frac{(B^r - B^R)^{-1}(B^r - B^R}{1-B} - AB^R\left\{R-1-r-(R-r-1)\frac{(1-B)}{1-(D+D')} + \frac{(1-B)}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right\}\right]$$

$$= \overline{P}_0(0)\left(c^c/c!\right)B^r\left[\frac{1}{(1-B)} - AB^R\left\{\frac{(R-1-r)(1-(D+D')-(R-r-1)(1-B)}{1-(D+D')} + \frac{(1-B)\sum_{n=1}^{R-r}(D+D')^n}{1-(D+D')}\right\}\right]$$

$$= \overline{P}_0(0)\left(c^c/c!\right)B^r\left[\frac{1}{(1-B)} - AB^R\left\{(R-1-r)\left[\frac{(1-(D+D')-(1-B)}{(1-(D+D'))}\right] + \frac{(1-B)}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right]\right.$$

$$= \overline{P}_0(0)\left(c^c/c!\right)B^r\left[\frac{1}{(1-B)} - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{(1-B)}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right]\right. \qquad (4.2.22))$$

(iv) The conditional probability that the system is in state - 0 when the system size lies between g and R is

$$\overline{P}(o/r \le n \le R) = \frac{\sum_{n=r}^{R-1}\overline{P}_n(0)}{\overline{P}(r \le n \le R)}$$

$$= \frac{\sum_{n=r}^{R-1} B^r(B^n - B^R)\frac{c^c}{c!}A\overline{P}_0(0)}{\overline{P}_0(0)\left(\frac{c^c}{c!}\right)B^r\left[\frac{1}{(1-B)} - AB^R\left\{\frac{(R-r)(B-(D+D')}{1-(D+D')} + \frac{1-B}{1-(D+D')}\sum_{n=1}^{R-r}(D+D)^n\right\}\right]}$$

$$= \frac{\frac{c^c}{c!}A\overline{P}_0(0)B^r\sum_{n=r}^{R-1}(B^n - B^R)}{\overline{P}_0(0)\left(\frac{c^c}{c!}\right)B^r\left[\frac{1}{(1-B)} - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{1-B}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right]\right.}$$

$$= \frac{(B^r - B^R)^{-1}\left[\sum_{n=r}^{R-1}(B^n - B^R)\right]}{\left[1/(1-B) - AB^R\left\{\frac{(R-r)(B-(D+D')}{1-(D+D')} + \frac{1-B}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right]\right.}$$

$$= \frac{(B^r - B^R)^{-1}\left[(B^r + B^{r+1} + \text{----} + B^{R-1}) - B^R(1+1+\text{----}+1)\right]}{\left[1/(1-B) - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{1-B}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right]\right.}$$

$$= \frac{(B^r - B^R)^{-1}\left[\left[\frac{B^r - B^R}{1-B}\right] - B^R(R-r)\right]}{\left[1/(1-B) - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{1-B}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right]\right.}$$

$$\overline{P}(0\ / r \le n \le R) = \frac{\frac{1}{(1-B)} - B^R.A.(R-r)}{\left[\frac{1}{(1-B)} - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{1-B}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right\}\right]}$$

(4.2.23)

(v) The conditional probability that the system is in state-1 and given that the system size is between r and R is obtained as.

$$\overline{P}(1/r \le n \le R) = \sum_{n=r+1}^{R}\overline{P}_n(1)/\overline{P}(r \le n \le R)$$

$$= \frac{\sum_{n=r+1}^{R}\overline{P}_{r+1}(1)\left(\frac{1-(D+D')^{n-r}}{1-(D+D')}\right)}{\overline{P}_0(0)\left(\frac{c^c}{c!}\right)B^r\left[\frac{1}{(1-B)} - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{1-B}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right\}\right]}$$

$$= \frac{B^{R+r}(1-B)\left(c^c/c!\right)A\overline{P}_0(0)\sum_{n=r+1}^{R}\left(\frac{1-(D+D')^{n-r}}{1-(D+D')}\right)}{\overline{P}_0(0)\left(c^c/c!\right)B^r\left[\frac{1}{(1-B)} - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{1-B}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right\}\right]}$$

$$\overline{P}\left(1/_{r \le n \le R}\right) =$$

$$\frac{B^R(1-B)\left(c^c/c!\right)(B^r - B^R)^{-1}B^r\overline{P}_0(0)\sum_{n=r+1}^{R}\frac{(1-(D+D')^{n-r}}{1-(D+D')}}{\overline{P}_0(0)\frac{c^c}{c!}B^r\left[\frac{1}{(1-B)} - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{(1-B)}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right\}\right]}$$

$$= \frac{\frac{B^R(1-B)(B^r - B^R)^{-1}}{1-(D+D')}\left[(R-r) - \sum_{n=1}^{R-r}(D+D^1)^n\right]}{\frac{1}{(1-B)} - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{1-B}{(1-(D+D'))}\sum_{n=1}^{R-r}(D+D')^n\right.}$$

$$= \frac{AB^R(1-B)/(1-(D+D'))\left[(R-r) - \sum_{n=1}^{R-r}(D+D')^n\right]}{\left[\frac{1}{(1-B)} - AB^R\left\{\frac{(R-r)(B-(D+D'))}{1-(D+D')} + \frac{1-B}{1-(D+D')}\sum_{n=1}^{R-r}(D+D')^n\right.\right.} \qquad (4.2.24)$$

(vi) Mean queue length :-

We calculate the expected number of customers in the system let Ls denote the over average number of customers in the system then we have

$Ls = Ls_0 + Ls_1$ (4.2.25)

where

$$L_{s_0} = \sum_{n=0}^{r} n\overline{p}_n(0) + \sum_{n=r+1}^{R-1} n\overline{P}_n(0) \qquad (4.2.26)$$

$$L_{s_1} = \sum_{n=r+1}^{R-1} n\overline{P}_n(1) + \sum_{n=R}^{\infty} n\overline{P}_n(1) \qquad (4.2.27)$$

$$L_{s_0} = \sum_{n=0}^{C-1} n\overline{P}_n(0) + \sum_{n=c}^{r} n\overline{P}_n(0) + \sum_{n=r+1}^{R-1} n\overline{P}_n(0)$$

$$=\sum_{n=0}^{C-1} n\left(\alpha_0{}^n/n!\right)\overline{P}_0(0)+\sum_{n=c}^{r} n.B^n\left(c^c\big/_{c!}\right)\overline{P}_0(0)+\sum_{n=r+1}^{R-1} n.B^r(B^n-B^R)\left(c^c\big/_{c!}\right)A\overline{P}_0(0)$$

$$=\sum_{n=0}^{C-1} n(\alpha_0{}^n/n!)\overline{p}_0(0)+\frac{c^c}{c!}\left[\sum_{n=c}^{r} n.B^n+AB^r\sum_{n=r+1}^{R-1} n.B^n-AB^{R+r}\sum_{n=r+1}^{R-1} n\right]\overline{p}_0(0)$$

$$(1)\sum_{n=c}^{r} n.B^n=\sum_{n=o}^{r-c}(n+c)B^{n+c}$$

$$S=cB^c+(c+1)B^{c+1}+(c+2)B^{c+2}+---\infty$$

$$Bs=c\,B^{c+1}+(c+1)\,B^{c+2}+--$$

$$(1-B)s=cB^c+B^{c+1}+B^{c+2}+---\infty$$

$$=cB^c+B^{c+1}(1+B+B^2+--\infty)$$

$$=cB^c+B^{c+1}\left(1\big/_{1-B}\right)$$

$$(1-B)\ s=c\ B^c+\frac{B^{c+1}}{1-B}$$

$$s=\frac{c\ B^c}{1-B}+\frac{B^{c+1}}{(1-B)^2}$$

$$s=\frac{(1-B)\ c\ B^c+B^{c+1}}{(1-B)^2}$$

$$=\frac{B^c\ (c-(c-1)B)}{(1-B)^2}$$

$$(II)\ A\ B^r\sum_{n=r+1}^{R-1} n\ B^n=A\ B^r\sum_{n=0}^{R-r-2}(n+r+1)\ B^{n+r+1}$$

$$S=(r+1)B^{r+1}+(r+2)B^{r+2}+---\infty$$

$$Bs=\qquad\qquad (r+1)B^{r+2}+---\infty$$

$$(1-B)S=(r+1)B^{r+1}+B^{r+2}+---\infty$$

$$=rB^{r+1}+B^{r+1}+B^{r+2}+---\infty$$

$$=rB^{r+1}+B^{r+1}(1+B+B^2+---\infty)$$

$$=rB^{r+1}+B^{r+1}\left(\frac{1}{1-B}\right)$$

$$s=\frac{r\,B^{r+1}}{1-B}+\frac{B^{r+1}}{(1-B)^2}=\frac{r\,B^{r+1}(1-B)+B^{r+1}}{(1-B)^2}$$

$$=\frac{r\,B^{r+1}-r\,B^{r+2}+B^{r+1}}{(1-B)^2}=\frac{B^{r+1}(\,(r+1)-r\,B)}{(1-B)^2}$$

$$=\frac{B^{R}(R-r\,B)}{(1-B)^2}\quad\text{where}\quad r+1=R$$

$$AB^{r}\sum_{n=r+1}^{R-1}n.B^{n}=AB^{r}\left[\frac{B^{R}(R-rB)}{(1-B)^2}\right]=AB^{R+r}\frac{(R-rB)}{(1-B)^2}$$

$$(III)AB^{R+r}\sum_{n=r+1}^{R+1}n=AB^{R+r}\left[(r+1)+(r+2)+--+(r+(R-1-r)\right]$$

$$=AB^{R+r}\left[r(1+1+1---+1)+(1+2+3+--+(R-1-r))\right]$$

$$=AB^{R+r}\left[r(R-r-1)+\frac{R-1-r}{2}[2+(R-1-r-1)\right]$$

$$=AB^{R+r}\left[(R-r-1)\left[r+\frac{R-r}{2}\right]\right]$$

$$=AB^{R+r}\left[(R-r-1)\left(\frac{2r+R-r}{2}\right)\right]$$

$$=AB^{R+r}\left[\left(\frac{(R-r-1)(R+r)}{2}\right)\right]$$

$$=AB^{R+r}(R+r)(R-r-1)/2$$

$$L_{S0}=\left[\frac{B^{c}(c-(c-1)B}{(1-B)^2}+\frac{AB^{R+r}(R-rB)}{(1-B)^2}-\frac{1}{2}AB^{R+r}(R+r)(R-r-1)\right]$$

$$\overline{P}_0(0)c^{c}\Big/_{c!}+\sum_{n=0}^{c-1}n\left(\alpha_0{}^{n}/n!\right)\overline{P}_0(0)\tag{4.2.27}$$

$$L_{S_1}=\sum_{n=r+1}^{R-1}nP_n(1)+\sum_{n=R}^{\infty}nP_n(1)$$

$$=\sum_{n=r+1}^{R-1}n\left[\frac{1-(D+D')^{n-r}}{1-(D+D')}\right]\overline{P}_{r+1}(1)+\sum_{n=R}^{\infty}n.(D+D')^{n}\left[\frac{(D+D')^{-R}-(D+D')^{-r}}{1-(D+D')}\right]\overline{P}_{r+1}(1)$$

$$=\overline{P}_{r+1}(1)\frac{1}{(1-(D+D'))}\left[\sum_{n=r+1}^{R-1}n-\sum_{n=r+1}^{R-1}n(D+D')^{n-r}+\left[(D+D')^{-R}-(D+D')^{-r}\right]\sum_{n=R}^{\infty}n.(D+D')^{n}\right]$$

Simplifying as in previous case

$$=\overline{P}_{r+1}(1)\frac{1}{[1-(D+D')]}\left[\frac{1}{2}(R+r)(R-r-1)-\sum_{n=r+1}^{R-1}n(D+D')^{n-r}+\left[(D+D')^{-R}-(D+D')^{-r}\right]\sum_{n=R}^{\infty}n(D+D')^{n}\right]$$

$$(I)\sum_{n=r+1}^{R-1}n(D+D')^{n-r}=(r+1)(D+D')+(r+2)(D+D')^{2}+(r+3)(D+D')^{3}+-----+(r+R-1-r)(D+D')^{R-r-1}$$

$$s=(r+1)\ (D+D')+(r+2)\ (D+D')^2+------+(r+(R-1-r)\ (D+D')^{R-r-1}$$

$$\frac{(D+D')\ s=(r+1)\ (D+D')^2+----+(r+R-2-r)\ (D+D')^{R-r-1}+(r+R-r-1)\ (D+D')^{R-r}}{(1-(D+D')\ s=(r+1)\ (D+D')+(D+D')^2---+(D+D')^{R-r-1}+(D+D')^{R-r}-R\ (D+D')^{R-r}}$$

$$=r\ (D+D')+(D+D')+(D+D')^2+--(D+D')^{R-r}-R\ (D+D')^{R-r}$$

$$=r\ (D+D')+\frac{(D+D')-(D+D')^{R-r-1}}{1-(D+D')}-R\ (D+D')^{R-r}$$

$$s=\frac{r\ (D+D')}{1-(D+D')}+\frac{(D+D')-(D+D')^{R-r+1}}{[1-(D+D')]^2}-\frac{R\ (D+D')^{R-r}}{1-(D+D')}$$

$$s=\frac{r\ (D+D')\ (1-D+D')\ +(D+D^1)-(D+D')^{R-r+1}-R\ (D+D')^{R-r}[(1-(D+D')]}{[1-(D+D')]^2}$$

$$=\frac{r\ (D+D')-r\ (D+D')^2+\ (D+D')-\ (D+D')^{R-r+1}-R\ (D+D')^{R-r}+R\ (D+D')^{R-r+1}}{[1-(D+D')]^2}$$

$$(II)\ \sum_{n=R}^{\infty} n\ (D+D')^n$$

$$s=R\ (D+D')^R+(R+1)\ (D+D')^{R+1}+\ (R+2)\ (D+D')^{R+2}+---$$

$$\frac{(D+D')\ s=R\ (D+D')^{R+1}+\ (R+1)\ (D+D')^{R+2}+--}{(1-D+D')\ s=R\ (D+D')^R+\ (D+D')^{R+1}+---\infty}$$

$$=R\ (D+D')^R+\frac{(D+D')^{R+1}}{1-(D+D')}$$

$$s=\frac{R\ (D+D')^R}{[1-(D+D')]}+\frac{(D+D')^{R+1}}{[1-(D+D')]^2}$$

$$=\frac{R\ (D+D')^R\ [1-(D+D')]\ +(D+D')^{R+1}}{[1-(D+D')]^2}$$

$$=\frac{R\ (D+D')^R\ -R\ (D+D^1)^{R+1}\ +(D+D')^{R+1}}{[1-(D+D')]^2}$$

$$L_{S_1}=\overline{P}_{r+1}(1)\frac{1}{[1-(D+D')]}\Big[\frac{1}{2}(R+r)(R-r-1)-r(D+D')+$$

$$r(D+D')^2-(D+D')+(D+D')^{R-r+1}+R(D+D')^{R-r}-R\,(D+D')^{R-r+1}+\left[(D+D')^{-R}-(D+D')^{-r}\right]*$$

$$\left\{\frac{R(D+D')^R-R(D+D')^{R+1}+(D+D')^{R+1}}{[1-(D+D')]^2}\right\}$$

$$=\overline{P}_{r+1}(1)\left[\frac{1}{1-(D+D')}\right]\Big[\frac{1}{2}(R+r)(R-r-1)+\{-r(D+D')+r(D+D')^2-(D+D^1)+(D+D')^{R-r+1}+R(D+D')^{R-r}$$

$$-R(D+D')^{R-r+1}+R-R(D+D')^{R-r}-R(D+D')+R(D+D')^{R-r+1}+(D+D^1)-(D+D')^{R-r+1}\ \}[1-(D+D')]^2\Big]$$

$$= \overline{P}_{r+1}(1)\left[\frac{1}{1-(D+D')}\right]\left[\frac{1}{2}(R+r)(R-r-1)+\left\{R-R(D+D')-r(D+D')+r(D+D')^2\right\}/[1-(D+D']^2\right\}$$

$$= \overline{P}_{r+1}(1)\left[\frac{1}{1-(D+D')}\right]\left[\frac{1}{2}(R+r)(R-r-1)+\left\{\frac{R(1-(D+D'))-r(D+D')(1-(D+D'))}{[1-(D+D')]^2}\right\}\right.$$

$$= \overline{P}_{r+1}(1)\left[\frac{1}{1-(D+D')}\right]\left[\frac{1}{2}(R+r)(R-r-1)+\left\{\frac{[R-r(D+D')][1-(D+D')]}{[1-(D+D')]^2}\right\}\right.$$

$$= \overline{P}_{r+1}(1)\left[\frac{1}{1-(D+D')}\right]\left[\frac{1}{2}(R+r)(R-r-1)+(R-r(D+D'))/(1-(D+D'))\right] \qquad (4.2.28)$$

$$L_S = L_{S0} + L_{S1}$$

$$= \left[\frac{B^c(c-(c-1)B)}{(1-B)^2} - (R-rB)\left(AB^{r+R}/(1-B)\right)\right.$$

$$-\frac{1}{2}AB^{R+r}(R+r)(R-r-1)\ \left]\overline{P}_0(0)\left(c^c/c!\right)+\sum_{n=0}^{C-1} n\left(\alpha_0{}^n/n!\right)\overline{P}_0(0)\right.$$

$$+\frac{\overline{P}_{r+1}}{(1-(D+D^1))}(1)\left[\frac{1}{2}(R+r)(R-r-1)+(R-r(D+D'))\ /(1-(D+D'))\right] \qquad (4.2.29)$$

(VII) Expected waiting time: using the Little formula the expected waiting time of the customer in the system is calculated as

$$W_s = L_s/\lambda \qquad (4.2.30)$$

Where λ is the actual mean arrival of the system and it is *given by*

$$\lambda = \lambda_0 P(0) + \lambda_1 P(1)$$

Substituting the values of $P(0)$ & $P(1)$ *from* equation (4.2.18) & (4.2.19)

$$\lambda = \lambda_0\left[\sum_{n=0}^{c-1}\frac{(BC)^n}{n!}\overline{P}_0(0)+\left(\frac{c^c}{c!}\right)\left[B^C\Big/(1-B)-A(R-r)B^{R+r}\right]\overline{P}_0(0)\right.$$

$$+\lambda_1\left[\overline{P}_{r+1}(1)(R-r)/(1-(D+D'))\right] \qquad (4.2.31)$$

Particular Cases

(1) *Single Server Poission queueing model with controllable arrival rates*

When $C = 1$ the $M/M/C$ interdependent queueing model with conbollable arrival rates reduce to $M/M/1$ interdepend queueing model with controlloble arrival rates.

Thus when $C = 1$ We have the following

For $1 \le n \le c-1$, $\overline{P}_n(0)$ is not possible.

For $1 \le n \le r$

$$\overline{P}_n(0) = \alpha_0^{\,n}\,\overline{P}_0(0)$$

For $r \le n \le R-1$

$$\overline{P}_n(0) = \alpha_0^{\,r}\left[\alpha_0^{\,n} - \alpha_0^{\,R}\right]A\overline{P}_0(0)[\text{Since when } C = 1,\ B = \alpha_0)$$

Where $A = \left[\alpha_0^{\,r} - \alpha_0^{\,R}\right]^{-1}$

Equationin (4.2.17) *gives*

$$\text{Similarly}\ \overline{P}_n(1) = \overline{P}_{r+1}(1)\left[\frac{1-(\alpha_1+\alpha_2)^{n-r}}{1-(\alpha_1+\alpha_2)}\right] \qquad r+1 \le n \le R\ \ (\text{Where } C = 1\ D = \alpha_1,\ D^1 = \alpha_2)$$

$$\overline{P}_n(1) = \overline{P}_{r+1}(1)\left[(\alpha_1+\alpha_2)^n\left\{\frac{(\alpha_1+\alpha_2)^{-R} - (\alpha_1+\alpha_2)^{-r}}{1-(\alpha_1+\alpha_2)}\right\}\right] \qquad n \ge R+1$$

Where $\overline{P}_{r+1}(1) = \alpha_0^{\,R+r}(1-\alpha_0)A\overline{P}_0(0)$

$$\overline{P}(0) = \left[\frac{\alpha_0}{1-\alpha_0} - A(R-r)\alpha_0^{R+r}\right]\overline{P}_0(0) \qquad \text{From equation} \qquad (4.2.18)$$

$$\overline{P}(1) = \left[(R-r)/(1-(\alpha_1+\alpha_2))\right]\overline{P}_{r+1}(1) \qquad \text{From equation} \qquad (4.2.19)$$

From Equation (4.2.29) *gives*

$$L(S) = \left[\alpha_0/(1-\alpha_0)^2 - (R - r\alpha_0)A\alpha_0^{R+r}/(1-\alpha_0)\right)$$

$$\left[\frac{1}{2}(R+r)(R-r-1) + (R - r\alpha_1 + \alpha_2)/(1-(\alpha_1+\alpha_2))\right]$$

(ii) *$M \mid M \mid C$ Model :-*

When $e = 0$ and $\lambda_0 = \lambda_1 = \lambda$ transient behaviour of the $M/M/C$ interdependent and queueing model with controllable arrival rates reduces to the usual $M/M/C$ Model [2].

Hence, When $e = o$, $s = 0$, $\lambda_0 = \lambda_1 = \lambda (r = R)$

equation (4.2.16) *gives*

$$\overline{P}_n(0) = \left(\frac{\lambda}{\mu}\right)^n\left(\frac{1}{n!}\right)\overline{P}_0(0) \qquad 1 \le n \le c-1$$

For $C \le n \le r$

$$\overline{P}_n(0) = (\lambda/\mu)^n\left(1/c^{n-c}c!\right)\overline{P}_0(0) \qquad \text{From } (4.2.16)$$

$\overline{P}_n(0)(r \le n < R-1)$ *and* $\overline{P}_n(1)(r+1 \le n \le R)$ *are not possible.*

From equation (4.2.17) *gives*

$$\overline{P}_n(1) = \left(\frac{\lambda}{c\mu}\right)^n \left[\frac{(\lambda/c\mu)^{-R} - (\lambda/c\mu)^{-r}}{1-(\lambda/c\mu)}\right]\overline{P}_{r+1}(1) \qquad n \ge R+1$$

Where

$$\overline{P}_{r+1}(1) = \left(\frac{\lambda}{c\mu}\right)^{R+r} (1-(\lambda/c\mu))\left(c^c/_{c!}\right)A\overline{P}_0(0)$$

and $A = \left[(\lambda/c\mu)^r - (\lambda/c\mu)^R\right]^{-1}$

i.e. $P_n(1) = (\lambda/\mu)^n \left(1/_{c^{n-c}c!}\right)\overline{P}_0(0) \qquad \text{n} \ge R+1$

Equation (4.2.20) *gives*

$$\left[\overline{P}_0(0)\right]^{-1} = \sum_{n=0}^{c-1}\frac{(BC)^n}{n!} + \frac{c^c}{c!}\left[\frac{B^c}{(1-B)} - A(R-r)B^{R+r}(B-(D+D')\ )/(1-(D+D')\ )\right]$$

$$\left[\overline{P}_0(0)\right]^{-1} = \sum_{n=0}^{c-1}\left((\lambda \mid \mu)^n/n!\right) + \frac{c^c}{c!}\left[\frac{(\lambda/\mu)^c}{1-\lambda/c\mu}\right]$$

$$= \sum_{n=0}^{c-1}\frac{(\lambda/\mu)^n}{n!} + \frac{c^c}{c!}(\lambda/\mu)^c\left[\frac{c\mu}{(c\mu-\lambda)}\right]$$

Equation (4.2.29) *gives*

$$L_S = \left[\frac{B^c(c-(c-1)B)}{(1-B)^2} - (R-rB)\left(AB^{R+r}/(1-B)\right)\right.$$

$$-\frac{1}{2}AB^{R+r}(R+r)(R-r-1)\Big]\overline{P}_0(0)\left(c^c/_{c!}\right)+$$

$$\sum_{n=0}^{c-1} n\left(\alpha_0^n/n!\right)\overline{P}_0(0) + \overline{P}_{r+1}(1)/(1-(D+D'))^*$$

$$\left[\frac{1}{2}(R+r)(R-r-1) + (R-r(D+D'))/(1-(D+D'))\right]$$

$$L_S = \sum_{n=0}^{c-1}\frac{n(\lambda/\mu)^n}{n!}\overline{P}_0(0) + \frac{c^c}{c!}\frac{\rho^c[c-\rho(c-1)]}{(1-\rho)^2}\overline{P}_0(0)$$

Where $\rho = \lambda/\mu c$

Thus the resuslts of usual M / M / C Model are obtained as a special case.

Chapter - 5

The Steady State Analysis of $M/G^X/1$ Retrial Queue With Multiple Vacations

INTRODUCTION

Retrial queueing system are characterized in a such a way that arriving customers who find the server busy join the retrial group and try again after random intervals for their request. In this chapter we consider customer's arrival one by one at a moment but service to all customer's is provided in bulk. A large number of applications related to transportation process such as by road transport, rail transport, airplane etc. involves bulk type of queue process.

Research on retrial queues with non-exponential retrial times is very limited. The first ever work reported in 1977, by Kapyrin [93] the author analyzed and derived an exact analytic solution for the M/G/1 retrial queue with general retrial time distribution. However, this methods was found to be incorrect and the results are erroneous as stated by Falin in 1986. Another account of the subject is the treatment of a G/M/S/O queueing loss system with retrial (Pour Babai, 1987), where returning customers are assumed to be a new form, dependent input process, input process, whose parameter are approximated from those of the overflow process from the service station. Retrial queues are widely used as mathematical models of several computer system, packet switching, network, sheared bus local area, network operating under the carrier sense and it is assumed that at every arrival epoch, a batch of primary customers arrives with a given probability.

If the server is busy or down due to on vacation at the arrival epoch, then all these customers join the group whereas if the server is free then one of the arriving customers begins his service and the others leave the service area and enter a group of blocked customer called orbit according with FCFS discipline.

Single server queues with vacation have been studied extensively in the past. These models arise naturally in telephone switching system, computer communication system etc. most of the analysis for retrial queues concern the exhaustive service schedule Artalezo [6] and gets the service policy Langaris [103]. Chaudhary and Templeton have provided a comprehensive review on bulk queues and their application Langaris and Moutzoukis [101] have considered a retrial queue with structured batch arrivals, preemptive resume priorities for a single vacation model. Yang and Templeton [148], Falin [53], Falin and Templeton [54] and Artalezo have presented a comprehensive survey and related work on retrial queue, queueing system with batch arrival are common in a number of real situations. In computer communication system, messages that are to be transmitted might have random number of packets. Comparable work on optimal control policies for batch arrival case is seldom found in the literature. This motivates us to develop a realistic model for queuing system with batch arrivals. Chaudhary and Templeton [33] have provided a comprehensive review on bulk queues and their application. Langaris and Moutzoukis [101] have considered a retrial queue with structured batch arrivals. Preemptive resume priorities for a single vacation model.

It is assume that a retrial time starts when the server completes a services and becomes free so that customer waits for completion of repair period. If the server is idle, an arriving customer (new or retrial) must starts the server, which takes negligible time. If the server starts successfully then the customer get services immediately, otherwise the repair to be done then the server commences service immediately and the customer must leave for the group and make a retrial at a later time.

The service of customers are independent random variable with common distribution B (x), density function b (x), Laplace – Stieltjes transform B*(q) and first two moments B_1 and B_2. Similarly the successive repair times are independent and identically distributed with probability distribution function H (x), density function h(x), Leplace – Stieltjes transform H* (q) and first two moments D_1 and D_2.

Further, the vacation time has distribution function v (x), density function V (x) Laplace – Stieltjes transform V* (q) and first two moment V_1 and V_2.

The state of the system at time t can be described by the Markov process {N (t); t^3 0} = {(C (t), X (t), X_0 (t), X_1 (t), X_2 (t), X_3 (t)); $t^3$0} where C (t) denotes the server state 0,1,2, or 3 according as the server being free, busy, down or on vacation respectively) and X (t) corresponds to the number of customers in the orbit at time t. If C (t) = 0 and X (t) > 0 then X_0 (t) represents the elapsed retrial time. If C (t) = 1, then x_1 (t) corresponds to the elapsed service time of the customer being served. If C (t) = 2 and X (t) > 0, then X_2 (t) represents the elapsed

repair time, if C (t) = 3 and X (t) > 0 then X_3 (t) represent the elapsed vacation time at time t. The function r (x), m (x), h (x) and n (x) are the conditional completion rates (at time x) for repeated attempts, for service, for repair and for vacation, respectively.

i.e.

$r(x) = a(x)\,[1-A(x)]^{-1}$ $m(x) = b(x)\,[1-B(x)]^{-1}$

$h(x) = h(x)\,[1-H(x)]^{-1}$ $n(x) = v(x)\,[1-V(x)]^{-1}$

Steady –state distribution :

we study the steady state distribution for the system under consideration. For the process $\{N(t);t=0\}$, we define the probability.

$P_n(x,t)dx = P\{c(t)=0, x(t)=n, x=\xi_0(t)<x+dx\}$ for $t=0, x=0 \;\&\; n=1$

$Q_n(x,t)dx = P\{c(t)=1, x(t)=n, x=\xi_1(t)<x+dx\}$ for $t=0, x=0 \;\&\; n=1$

$R_n(x,t)dx = P\{c(t)=2, x(t)=n, x=\xi_2(t)<x+dx\}$ for $t=0, x=0 \;\&\; n=0$

$V_n(x,t)dx = P\{c(t)=3, x(t)=n, x=\xi_3(t)<x+dx\}$ for $t=0, x=0 \;\&\; n=0$

By supplementary variable technique we obtain the following system of equations that govern the dynamics of the system behavior.

$$\frac{\partial P_n(x,t)}{\partial t}+\frac{\partial P_n(x,t)}{\partial x}=-(\lambda+r(x))P_n(x,t),\ \mathrm{n}=1,2,3.... \tag{5.2.1}$$

$$\frac{\partial Q_0(x,t)}{\partial t}+\frac{\partial Q_0(x,t)}{\partial x}=-(\lambda+\mu(x))Q_0(x,t)+\mu(x)\sum_{i=1}^{k}C_kQ_i(x,t) \tag{5.2.2}$$

$$\frac{\partial Q_n(x,t)}{\partial t}+\frac{\partial Q_n(x,t)}{\partial x}=-(\lambda+\mu(x))Q_n(x,t)+\mu(x)\ C_kQ_{n+k}(x,t),\ \mathrm{n}=1,2,3... \tag{5.2.3}$$

$$\frac{\partial R_1(x,t)}{\partial t}+\frac{\partial R_1(x,t)}{\partial x}=-(\lambda+\eta(x))R_1(x,t)+\eta(x)\sum_{i=1}^{k}C_kR_i(x,t) \tag{5.2.4}$$

$$\frac{\partial R_n(x,t)}{\partial t}+\frac{\partial R_n(x,t)}{\partial x}=-(\lambda+\eta(x))R_n(x,t)+\eta(x)\ C_kR_{n+k}(x,t),\ \mathrm{n}=2,3,4... \tag{5.2.5}$$

$$\frac{\partial V_0(x,t)}{\partial t}+\frac{\partial V_0(x,t)}{\partial x}=-(\lambda+V(x))V_0(x,t)+V(x)\sum_{i=1}^{k}C_kV_i(x,t) \tag{5.2.6}$$

$$\frac{\partial V_n(x,t)}{\partial t}+\frac{\partial V_n(x,t)}{\partial x}=-(\lambda+v(x))V_n(x,t)\ V(x)\ C_kV_{n+k}(x,t),\ \mathrm{n}=1,2,3...... \tag{5.2.7}$$

The boundary conditions are :-

$$\mathrm{P_n}(0,t)=p\int_0^{\infty}Q_n(x,t)\mu(x)dx+\int_0^{\infty}R_n(x,t)\eta(x)dx+\int_0^{\infty}V_n(x,t)V(x)dx,\ \mathrm{n}=1,2,3... \tag{5.2.8}$$

$$Q_0(0,t) = \alpha \int_0^\infty P_1(x,t) r(x) dx \quad (5.2.9)$$

$$Q_n(0,t) = \alpha \int_0^\infty P_{n+1}(x,t) r(x) dx + \alpha\lambda \int_0^\infty C_k P_{n+k+1}(x,t) dx, \quad n = 1,2,3 \quad (5.2.10)$$

$$R_1(0,t) = \overline{\alpha} \int_0^\infty P_1(x,t) r(x) dx \quad (5.2.11)$$

$$R_n(0,t) = \overline{\alpha} \int_0^\infty P_n(x,t) r(x) dx + \overline{\alpha}\lambda \int_0^\infty C_k P_{n+k+1}(x,t) dx, \quad n=2,3,4..... \quad (5.2.12)$$

$$V_0(0,t) = \int_0^\infty Q_0(x,t)\mu(x) dx + \int_0^\infty V_0(x,t) v(x) dx \quad (5.2.13)$$

$$V_n(0,t) = \int_0^\infty Q_n(x,t)\mu(x) dx \quad n = 1,2,3... \quad (5.2.14)$$

We assume that condition $\alpha\lambda\overline{c}_1(B_1 + qV_1) + \overline{\alpha}(1 + \overline{c}_1\lambda D_1) + \overline{c}_1(1 - A^*(\lambda)) < 1$ is fulfilled so that we can set the limiting densities.

$P_n(x) = Lim_{t\to\infty} P_n(x,t)$ for x = 0 & n = 1

$Q_n(x) = Lim_{t\to\infty} Q_n(x,t)$ for x = 0 & n = 0

$R_n(x) = Lim_{t\to\infty} R_n(x,t)$ for x = 0 & n = 1

$V_n(x) = Lim_{t\to\infty} V_n(x,t)$ for x = 0 & n = 0

letting $t\to\infty$ in equations (5.2.1) – (5.2.14) we get

$$\frac{dP_n(x)}{dx} = -(\lambda + r(x))P_n(x) \quad n = 1,2,3.... \quad (5.2.15)$$

$$\frac{dQ_0(x)}{dx} = -(\lambda + \mu(x))Q_0(x) + \mu(x)\sum_{i=1}^{K} c_K Q_i(x) \quad (5.2.16)$$

$$\frac{dQ_n(x)}{dx} = -(\lambda + \mu(x))Q_n(x) + \mu(x)c_K Q_{n+K}(x) \qquad n = 1,2,3\ldots \tag{5.2.17}$$

$$\frac{dR_1(x)}{dx} = -(\lambda + \eta(x))R_1(x) + \eta(x)\sum_{i=1}^{K} c_K R_i(x) \tag{5.2.18}$$

$$\frac{dR_n(x)}{dx} = -(\lambda + \eta(x))R_n(x) + \eta(x)c_K R_{n+K}(x) \qquad n = 2,3\ 4\ldots \tag{5.2.19}$$

$$\frac{dV_0(x)}{dx} = -(\lambda + v(x))V_0(x) + v(x)\sum_{i=1}^{K} c_K V_i(x) \tag{5.2.20}$$

$$\frac{dV_n(x)}{dx} = -(\lambda + v(x))V_n(x) + v(x)c_K V_{n+K}(x) \quad n = 1,2,3\ldots.. \tag{5.2.21}$$

The steady state boundary conditions are :

$$P_n(0) = P\int_0^\infty Q_n(x)\mu(x)dx + \int_0^\infty R_n(x)\eta(x)dx + \int_0^\infty V_n(x)V(x)dx, \quad n = 1,2,3\ldots \tag{5.2.22}$$

$$Q_0(0) = \alpha\int_0^\infty P_1(x)r(x)dx \tag{5.2.23}$$

$$Q_n(0) = \alpha\int_0^\infty P_{n+1}(x)r(x)dx + \alpha\lambda\int_0^\infty C_k P_{n+k+1}(x)dx, \qquad n = 1,2,3\ldots. \tag{5.2.24}$$

$$R_1(0) = \overline{\alpha}\int_0^\infty P_1(x)r(x)dx \tag{5.2.25}$$

$$R_n(0) = \overline{\alpha}\int_0^\infty P_n(x)r(x)dx + \overline{\alpha}\lambda\int_0^\infty C_k P_{n+k}(x)dx, \qquad n=2,3,4\ldots.. \tag{5.2.26}$$

$$V_0(0) = \int_0^\infty Q_0(x)\mu(x)dx + \int_0^\infty V_0(x)V(x)dx \tag{5.2.27}$$

$$V_n(0) = q\int_0^\infty Q_n(x)\mu(x)dx, \qquad n = 1,2,3\ldots. \tag{5.2.28}$$

And the normalizing conditions is

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} P_n(x)dx + \sum_{n=0}^{\infty} \int_{0}^{\infty} Q_n(x)dx + \sum_{n=1}^{\infty} \int_{0}^{\infty} R_n(x)dx + \sum_{n=0}^{\infty} \int_{0}^{\infty} V_n(x)dx = 1 \quad (5.2.29)$$

We define the probability generating functions

$$P(x,z) = \sum_{n=1}^{\infty} P_n(x)\, z^n \qquad Q(x,z) = \sum_{n=0}^{\infty} Q_n(x)\, z^n$$

$$R(x,z) = \sum_{n=1}^{\infty} R_n(x)\, z^n \qquad V(x,z) = \sum_{n=0}^{\infty} V_n(x)\, z^n$$

The following theorem discusses the steady state distribution of the system.

Theorem 1 :- If the system is under steady state that is

$$\alpha\lambda\bar{c}_1(B_1 + qV_1) + \bar{\alpha}(1 + \bar{c}_1\lambda D_1) + \bar{c}_1(1 - A^*(\lambda)) < 1$$

then the steady-state distribution of N(t) = $\{c(t), x(t), t = 0\}$ is obtained as

$$P(x,Z) = \frac{V_0(0)(P + qV^*(\lambda))\left[1 - V^*(\lambda(1 - c(Z)Z^{-2K}))\right]e^{-\lambda x}[1 - A(x)]}{D(Z)} \quad (5.2.30)$$

$$Q(x,Z) = \frac{V_0(0)\left\{\alpha(P + qV^*(\lambda))\left(1 - V^*(\lambda(1 - c(Z)Z^{-2K}))\right)\left(c(Z)Z^{-2K} + A^*(\lambda)(1 - c(Z)Z^{-2K})\right)e^{-\lambda x}[1 - B(x)]\right\}}{D(Z)}$$

(5.2.31)

$$R(x,Z) = \frac{V_0(0)\left\{\bar{\alpha}Z(P + qV^*(\lambda))\left(1 - V^*(\lambda(1 - c(Z)Z^{-2K}))\right)\left(c(Z)Z^{-2K} + A^*(\lambda)(1 - c(Z)Z^{-2K})\right)e^{-\lambda x}[1 - H(x)]\right\}}{D(Z)}$$

(5.2.32)

$$V(x,Z) = \frac{V_0(0)(P + qV^*(\lambda))\left\{\left[c(Z)Z^{-2K} + A^*(\lambda)(1 - c(Z)Z^{-2K})\right]\left[\alpha B^*(\lambda(1 - c(Z)Z^{-2K})\right] + \bar{\alpha}ZH^*(\lambda(1 - c(Z)Z^{-2K})) - Z\right\}e^{-\lambda\lambda x}[1 - V(x)]}{D(Z)}$$

(5.2.33)

Where D(Z) =

$$\left\{c(Z)Z^{-2K} + A^*(\lambda)(1 - c(Z)Z^{-2K})\right\}\left\{\left[\alpha B^*(\lambda(1 - c(Z)Z^{-2K}))\right]\left\{P + qV^*(\lambda(1 - c(Z)Z^{-2K}))\right\} + \bar{\alpha}ZH^*(\lambda(1 - c(Z)Z^{-2K}))\right\} - Z$$

and the probability $V_0(0)$ is to be determined from the normalizing condition.

Proof : -

Multiplying both sides of equation (5.2.15) by z^n and summing with respect to n =1,2,3.... we obtain

$$\sum_{n=1}^{\infty}\frac{dP_n(x)}{dx}z^n = -\sum_{n=1}^{\infty}(\lambda+r(x))P_n(x)\,z^n$$

$$\frac{\partial P(x,z)}{\partial x} = -\ (\lambda+r(x))P\ (x,z) \qquad (5.2.34)$$

Now multiplying equations (5.2.16) & (5.2.17) by z^n and summing over n= 0,1,2. ∞ yields.

$$\frac{dQ_0(x)}{dx} = -(\lambda+\mu(x)\ \ Q_0(x) + \mu(x)\sum_{i=1}^{k}C_kQ_n(x) \qquad (5.2.16)$$

$$\frac{dQ_n(x)}{dx} = -(\lambda+\mu(x))\ Q_n(x) + \mu(x)C_k\ Q_{n+k}(x), \qquad n = 1,2,3..... \qquad (5.2.17)$$

$$\sum_{n=0}^{\infty}\frac{dQ_0(x)}{dx}z^n + \sum_{n=0}^{\infty}\frac{dQ_n(x)}{dx}z^n = -(\lambda+\mu(x))\left[\sum_{n=0}^{\infty}Q_0(x)z^n + \sum_{n=0}^{\infty}Q_n(x)z^n\right] +$$

$$\mu(x)\left[\sum_{n=0}^{\infty}\left[\sum_{i=1}^{k}C_kQ_i(x)z^n + C_kQ_{n+k}(x)z^n\right]\right]$$

$$\frac{\partial Q(x,z)}{\partial x} = -\ (\lambda+\mu(x))\ Q(x,z) +$$

$$\mu(x)\left[\sum_{n=0}^{\infty}(\ C_k(Q_1(x)+Q_2(x)+....+Q_{k-1}(x)+Q_k(x)+Q_{n+k}(x)\ )z^n\right]$$

$$\frac{\partial Q(x,z)}{\partial x} = -\ (\lambda+\mu(x))\ Q(x,z) +$$

$$\mu(x)\left[C_k\{Q_1(x)+Q_2(x)+.....+Q_{k-1}(x)+Q_k(x)+Q_{k+1}(x)+.....\infty\ \}z^n\right]$$

$$\frac{\partial Q(x,z)}{\partial x} = -\ (\lambda+\mu(x))\ Q(x,z) + \mu(x)\sum_{n=0}^{\infty}C_k\left(\sum_{k=1}^{\infty}C_kQ_{k+n}(x)Z^n\right)$$

$$= -\ (\lambda+\mu(x))\ Q(x,z) + \mu(x)\sum_{n=0}^{\infty}Q_{k+n}(x)Z^{n-K}\ \sum_{k=1}^{\infty}C_kZ^k$$

$$= - (\lambda + \mu(x))\ Q(x, z) + \ \mu(x)C(Z)\sum_{n=0}^{\infty} Q_{k+n}(x)Z^{n-k}$$

$$= - (\lambda + \mu(x))\ Q(x, z) + \mu(x)C(Z)\ Z^{-2k}Q(x,z)$$

$$\frac{\partial Q(x,z)}{\partial x} = - (\lambda + \mu(x))\ Q(x, z) + \mu(x)C(Z)\ Z^{-2k}Q(x,z)$$

$$\frac{\partial Q(x,z)}{\partial x} = -\left[\lambda + \mu(x)\left\{1 - C(Z)Z^{-2k}\ \right\}\right]Q(x,z) \qquad (5.2.35)$$

Now multiplying equations (5.2.18) & (5.2.19) by Z^n and summing over $n = 0,1,2.....$ yields

$$\frac{dR_1(x)}{dx} = -(\lambda + \eta(x))R_1(x) + \eta(x)\sum_{i=1}^{k} C_k R_1(x) \qquad (5.2.18)$$

$$\frac{dR_n(x)}{dx} = -(\lambda + \eta(x))\ R_n(x) + \eta(x)C_k R_{n+k}(x), \qquad n = 2,3,4....... \qquad (5.2.19)$$

$$\sum_{n=0}^{\infty}\frac{dR_1(x)}{dx}Z^n + \sum_{n=0}^{\infty}\frac{dR_n(x)}{dx}Z^n = -(\lambda + \eta(x))\left[\sum_{n=0}^{\infty}R_1(x)Z^n + \sum_{n=0}^{\infty}R_n(x)Z^n\right] +$$

$$\eta(x)\left[\sum_{n=0}^{\infty}\left[\sum_{i=1}^{k}C_k R_i(x)Z^n + C_k R_{n+k}(x)Z^n\right]\right]$$

$$\frac{\partial R(x,z)}{\partial x} = -(\lambda + \eta(x))\ R(x, z) + \ \eta(x)\left[\sum_{n=0}^{\infty}C_k(R_1(x) + R_2(x) + ... + R_k(x) + R_{n+k}(x))Z^n\right]$$

$$\frac{\partial R(x,z)}{\partial x} = -(\lambda + \eta(x))\ R(x, z) + \ \eta(x)\left[C_k(R_1(x) + R_2(x) + ... + R_k(x) + R_{k+1}(x) + ...\infty)Z^n\right]$$

$$= -(\lambda + \eta(x))\ R(x, z) + \ \eta(x)\sum_{n=0}^{\infty}C_k\left[\sum_{k=1}^{\infty}R_{n+k}(x)\right]Z^n$$

$$= \ -(\lambda + \eta(x))\ R(x, z) + \ \eta(x)\sum_{n=0}^{\infty}R_{n+k}(x)Z^{n-k}\sum_{k=1}^{\infty}C_k Z^k$$

$$= -(\lambda + \eta(x))\ R(x, z) + \ \eta(x)C(Z)Z^{-2k}R(x,z)$$

$$\frac{\partial R(x,z)}{\partial x} = -\left[\lambda + \eta(x)\left\{1 - C(Z)Z^{-2k}\ \right\}\right]R(x,z) \qquad (5.2.36)$$

Again now multiplying equation (5.2.20) & (5.2.21) by Z^n and summing over $n = 0,1,2.....$yields

$$\frac{dV_0(x)}{dx} = -(\lambda + V(x))\, V_0(x) + V(x) \sum_{i=1}^{k} C_k V_i(x) \tag{5.2.20}$$

$$\frac{dV_n(x)}{dx} = -(\lambda + V(x))\, V_n(x) + V(x)\, C_k V_{n+k}(x) \tag{5.2.21}$$

$$\frac{\partial V(x,z)}{\partial x} = -\ (\lambda + V(x))\, V(x,z) + V(x) \sum_{n=0}^{\infty} C_k \left[\sum_{k=1}^{\infty} R_{n+k}(x) \right] Z^n$$

$$= -(\lambda + V(x))\, V(x,z) + V(x)\ C(Z)\, Z^{-2k}\, R(x,z)$$

$$\frac{\partial V(x,z)}{\partial x} = -\ \left[\lambda + V(x)\{1 - C(Z)Z^{-2k}\ \}\right] V(x,z) \tag{5.2.37}$$

$$P_n 0) = p \int_0^{\infty} Q_n(x)\mu(x)dx + \int_0^{\infty} R_n(x)\eta(x)dx + \int_0^{\infty} V_n(x)V(x)dx,$$

$$\sum_{n=1}^{\infty} P_n(0)Z^n = \sum_{n=0}^{\infty} \left[P \int_0^{\infty} Q_n(x)\mu(x)dx \right] Z^n + \sum_{n=1}^{\infty} \int_0^{\infty} [R_n(x)\eta(x)dx] Z^n + \sum_{n=0}^{\infty} \left[\int_0^{\infty} V_n(x)V(x)dx \right] Z^n$$

$$P(0,Z) = P \int_0^{\infty} Q(x,z)\mu(x)dx + \int_0^{\infty} R(x,z)\eta(x)dx + \int_0^{\infty} V(x,z)V(x)dx - p \int_0^{\infty} Q_0(x)\mu(x)dx - \int_0^{\infty} V_0(x)V(x)dx \tag{5.2.38}$$

From (5.2.23) & (5.2.24)

$$Q_0(0) = \alpha \int_0^{\infty} P_1(x)r(x)dx \tag{5.2.23}$$

$$Q_n(0) = \alpha \int_0^{\infty} P_{n+1}(x)r(x)dx + \alpha\lambda \int_0^{\infty} C_k P_{n+k+1}(x)dx,\ n=1,2,3\ldots \tag{5.2.24}$$

$$\sum_{n=0}^{\infty} Q_n(0)Z^n = \sum_{n=1}^{\infty}\ \alpha \int_0^{\infty} P_{n+1}(x)r(x)dxz^n + \alpha \frac{\lambda z^{-2k}}{z} \sum_{n=1}^{\infty} C_k Z^k \int_0^{\infty} P_{n+k+1} Z^{n+k+1} dx$$

$$Q(0,z) = \frac{\alpha}{z}\int_0^\infty P(x,z)r(x)dx + \frac{\alpha\lambda}{z}Z^{-2k}\int_0^\infty C(z)P(x,z)dx.c(z)$$

$$= \frac{\alpha}{z}\left[\int_0^\infty P(x,z)r(x)dx + \lambda c(z)z^{-2k}\int_0^\infty P(x,z)dx\right] \tag{5.2.39}$$

From (5.2.25) & (5.2.26)

$$R_1(0) = \overline{\alpha}\int_0^\infty P_1(x)r(x)dx \tag{5.2.25}$$

$$R_n(0) = \overline{\alpha}\int_0^\infty P_n(x)r(x)dx + \overline{\alpha}\lambda\int_0^\infty C_k P_{n+k}(x)dx \tag{5.2.26}$$

$$\sum_{n=1}^\infty R_n(0)Z^n = \overline{\alpha}\sum_{n=1}^\infty\int_0^\infty P_n(x)Z^n r(x)dx + \overline{\alpha}\lambda z^{-2k}\sum_{n=1}^\infty C_k z^k\int_0^\infty P_{n+k}(x)dxz^{n+k}$$

$$R(0,z) = \overline{\alpha}\int_0^\infty P(x,z)r(x)dx + \overline{\alpha}\lambda C(z)Z^{-2k}\int_0^\infty P(x,z)dx \tag{5.2.40}$$

From equation (5.2.27) & (5.2.28)

$$V_0(0) = \int_0^\infty Q_0(x)\mu(x)dx + \int_0^\infty V_0(x)V(x)dx \tag{5.2.27}$$

$$V_n(0) = q\int_0^\infty Q_n(x)\mu(x)dx, \qquad n = 1,2,3.... \tag{5.2.28}$$

$$\sum_{n=0}^\infty V_n(0)z^n = q\sum_{n=1}^\infty\int_0^\infty Q_n(x)Z^n\mu(x)dx + \int_0^\infty V_0(x)V(x)dx + \int_0^\infty Q_0(x)\mu(x)dx$$

$$\sum_{n=0}^\infty V_n(0)z^n = q\int_0^\infty Q_n(x)\mu(x)dx - q\int_0^\infty Q_0(x)\mu(x)dx + \int_0^\infty V_0(x)V(x)dx + \int_0^\infty Q_0(x)\mu(x)dx$$

$$V(0,Z) = q\int_0^\infty Q(x,z)\mu(x)dx + \int_0^\infty V_0(x)V(x)dx + (1-q)\int_0^\infty Q_0(x)\mu(x)dx$$

$$V(0,Z) = q\int_0^{\infty} Q(x,z)\mu(x)dx + \int_0^{\infty} V_0(x)V(x)dx + P\int_0^{\infty} Q_0(x)\mu(x)dx \tag{5.2.41}$$

Solving the partial differential equations (5.2.34) – (5.2.37)

$$P(x,z) = P(0,Z)\ e^{-\lambda x - \int_0^x r(u)du} \tag{5.2.42}$$

$$Q(x,z) = Q(0,Z)\ e^{-\lambda x - (1-c(z)Z^{-2k})\int_0^x \mu(u)du} \tag{5.2.43}$$

$$R(x,z) = R(0,z)\ e^{-\lambda x - (1-c(z)Z^{-2k})\int_0^x \eta(u)du} \tag{5.2.44}$$

$$V(x,z) = V(0,Z)\ e^{-\lambda x - (1-c(z)Z^{-2k})\int_0^x V(u)du} \tag{5.2.45}$$

From equation (5.2.16) –(5.2.20) we get

$$\frac{dQ_0(x)}{dx} = -(\lambda + \mu(x)Q_0(x) + \mu(x)\sum_{i=1}^{k} C_k Q_i(x) \tag{5.2.16}$$

The boundary conditions when x=0 so gives $Q_0(x) = Q_0(0)$ So the differential equation (5.2.16) on solution gives

$$Q_0(x) = Q_0(0)\ e^{-\lambda x - (1-C_k)\int_0^x \mu(u)du} \tag{5.2.46}$$

$$\frac{dV_0(x)}{dx} = -(\lambda + V(x)V_0(x) + V(x)\sum_{i=1}^{k} C_k V_i(x) \tag{5.2.20}$$

$$V_0(x) = V_0(0)\ e^{-\lambda x - (1-C_k)\int_0^x V(u)du} \tag{5.2.47}$$

From Equation (5.2.46)

$$\int_0^{\infty} Q_0(x)\mu(x)dx = \int_0^{\infty} e^{-\lambda x - (1-C_k)\int_0^x \mu(u)du}\ Q_0(0)\mu(x)d\,x$$

$$\int_0^{\infty} Q_0(x)\mu(x)dx = Q_0(0)\int_0^{\infty} e^{-\lambda x - (1-C_k)\int_0^x \mu(u)du}\ \mu(x)dx$$

$$Q_0(0) = \frac{1}{B^*(\lambda)} \int_0^\infty Q_0(x)\mu(x)dx \tag{5.2.48}$$

Where $B^*(\lambda) = \int_0^\infty \mu(x) e^{-\lambda x - (1-C_k)\int_0^x \mu(u)du} dx$

Similarly

$$V_0(0) = \frac{1}{V^*(\lambda)} \int_0^\infty V_0(x)V(x)dx$$

Where $V^*(\lambda) = \int_0^\infty V(x) e^{-\lambda x - (1-C_k)\int_0^x V(u)du} dx$ (5.2.49)

Using (5.2.48) and (5.2.49) in (5.2.27) we obtain

$$V_0(0) = \int_0^\infty Q_0(x)\mu(x)dx + \int_0^\infty V_0(x)V(x)dx \tag{5.2.27}$$

$$V_0(0) = Q_0(0)\ B^*(\lambda) + V_0(0)\ V^*(\lambda)$$

$$Q_0(0)\ B^*(\lambda) = V_0(0)\,[1 - V^*(\lambda)] \tag{5.2.50}$$

Now substituting (5.2.42) in (5.2.39) – (5.2.40) and on simplification

$$Q(0,Z) = \frac{\alpha}{z}[\int_0^\infty P(x,z)r(x)dx\ +\ \lambda C(z)Z^{-2k}\int_0^\infty P(x,z)dx \tag{5.2.39}$$

$$= \frac{\alpha}{z}[\int_0^\infty P(0,z)e^{-\lambda x - \int_0^x r(u)du}\ r(x)dx + \lambda c(z)Z^{-2k}\int_0^\infty P(0,Z)\ e^{-\lambda x - \int_0^x r(u)du}\ dx]$$

$$= \frac{\alpha}{z}P(0,Z)\left[\int_0^\infty e^{-\lambda x - \int_0^x r(u)du}\ r(x)dx + \lambda c(z)Z^{-2k}\int_0^\infty e^{-\lambda x - \int_0^x r(u)du}\ dx\right] \tag{A}$$

$$\int_0^\infty \lambda e^{-\lambda x - \int_0^x r(u)du}\ dx = \int_0^\infty \lambda e^{-\lambda x} e^{-\int_0^x r(u)du}\ dx$$

$$= \left[e^{-\int_0^x r(u)du} \int \lambda e^{-\lambda x} dx \right]_0^\infty - \int_0^\infty \left[\left(\frac{d}{dx} e^{-\int_0^x r(u)du} \right) \int \lambda e^{-\lambda x} dx \, dx \right]$$

$$= \left[e^{-\int_0^x r(u)du} e^{-\lambda x} \right]_0^\infty - \int_0^\infty e^{-\lambda x} e^{-\int_0^x r(u)du} \, \mathrm{r(x)\ dx}$$

$$= 1 - \int_0^\infty e^{-\lambda x - \int_0^x r(u)d(u)} \, \mathrm{r(x)\ dx}$$

Putting in (A)

$$Q(0, Z) = \frac{\alpha}{z} P(0,Z) \left[\int_0^\infty e^{-\lambda x - \int_0^x r(u)du} r(x)dx \right] + C(z)Z^{-2k} \left(1 - \int_0^\infty e^{-\lambda x - \int_0^x r(u)du} r(x)dx \right)$$

$$= \frac{\alpha}{z} P(0,Z) \left[A^*(\lambda) + c(z)Z^{-2k} (1 - A^*(\lambda) \right]$$

$$= \frac{\alpha}{z} P(0,Z) \left[c(z)Z^{-2k} + A^*(\lambda)(1 - c(z)Z^{-2k}) \right]$$

$$Q(0,Z) = \frac{\alpha}{z} P(0,Z) \left[c(z)Z^{-2k} + A^*(\lambda)(1 - c(z)Z^{-2k}) \right] \tag{5.2.51}$$

Now substituting (5.2.42) in (5.2.39) – (5.2.40) and on simplification

$$R(0,Z) = \overline{\alpha} \int_0^\infty P(x,z)r(x)dx + \overline{\alpha}\lambda c(z)Z^{-2k} \int_0^\infty P(x,z)dx$$

$$= \overline{\alpha} \int_0^\infty P(0,Z) e^{-\lambda x - \int_0^x r(u)du} r(x)dx + \overline{\alpha}\lambda c(z)Z^{-2k} \int_0^\infty P(0,Z) e^{-\lambda x - \int_0^x r(u)du} dx$$

$$= \overline{\alpha} \, P(0,Z) \left[\int_0^\infty e^{-\lambda x - \int_0^x r(u)du} r(x)dx + \lambda c(z)Z^{-2k} \int_0^\infty e^{-\lambda x - \int_0^x r(u)du} dx \right.$$

$$= \bar{\alpha}\ \mathrm{P(0,Z)} \left[\int_0^\infty e^{-\lambda x - \int_0^x r(u)du} r(x)dx + c(z)Z^{-2k} \left(1 - \int_0^\infty e^{-\lambda x - \int_0^x r(u)du} r(x)dx \right) \right.$$

$$= \bar{\alpha}\ \mathrm{P(0,Z)} \left[A^*(\lambda) + c(z)Z^{-2k}(1 - A^*(\lambda)) \right]$$

$$= \bar{\alpha}\ \mathrm{P(0,Z)} \left[c(z)Z^{-2k} + A^*(\lambda)(1 - c(z)Z^{-2k}) \right] \qquad (5.2.52)$$

Combining (5.2.41)-(5.2.49) and (5.2.48)-(5.2.51) after some mathematical manipulation, we get

$$\mathrm{V(0,Z)} = q\int_0^\infty Q(x,z)\mu(x)dx + \int_0^\infty V_0(x)V(x)dx + q\int_0^\infty Q_0(x)\mu(x)dx$$

$$\mathrm{P(x,z)} = \mathrm{P(0,Z)}\ e^{-\lambda x - \int_0^x r(u)du}$$

$$\mathrm{Q(x,z)} = \mathrm{Q(0,Z)}\ e^{-\lambda x - (1 - c(z)Z^{-2k})} \int_0^x \mu(u)du$$

$$\mathrm{Q_0(0)} = \frac{1}{B^*(\lambda)} \int_0^\infty Q_0(x)\mu(x)dx$$

$$\mathrm{V_0(0)} = \frac{1}{V^*(\lambda)} \int_0^\infty V_0(x)V(x)dx$$

$$\mathrm{Q(0,Z)} = \frac{\alpha}{Z} P(0,Z) \left[C(Z)Z^{-2k} + A^*(\lambda)(1 - C(Z)Z^{-2k}) \right]$$

Putting in (5.2.41), we get

$$\mathrm{V(0,Z)} = q\int_0^\infty Q(x,z)\mu(x)dx + \int_0^\infty V_0(x)V(x)dx + p\int_0^\infty Q_0(x)\mu(x)dx$$

$$= q\int_0^\infty Q(0,Z)e^{-\lambda x - (1 - c(Z)Z^{-2k})\int_0^x \mu(u)du} \mu(x)dx + V_0(0)V^*(\lambda) + PQ_0(0)B^*(\lambda)$$

$$=\frac{q\alpha}{Z}P(0,Z)\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]\int_0^\infty\left[\mu(x)e^{-\lambda x-(1-C(Z)Z^{-2k})\int_0^x\mu(u)du}dx\right]+$$

$$V_0(0)V^*(\lambda)+P\left[V_0(0)(1-V^*(\lambda))\right]$$

$$=\frac{q\alpha}{Z}P(0,Z)\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]B^*(\lambda(1-C(Z)Z^{-2k}))+$$

$$V_0(0)V^*(\lambda)(1-P)+PV_0(0)$$

$$=\frac{q\alpha}{Z}P(0,Z)\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]B^*(\lambda(1-C(Z)Z^{-2k}))+$$

$$V_0(0)\left[P+qV^*(\lambda)\right] \qquad (5.2.53)$$

Similarly substituting (5.2.43)-(5.2.49) in (5.2.38) we get

$$\mathrm{P}(0,Z)=P\int_0^\infty Q(x,z)\mu(x)dx+\int_0^\infty R(x,Z)\eta(x)dx+\int_0^\infty V(x,Z)V(x)dx-P\int_0^\infty Q_0(x)\mu(x)dx-$$

$$\int_0^\infty V_0(x)V(x)dx \qquad (5.2.38)$$

$$=P\int_0^\infty Q(0,Z)e^{-\lambda x-(1-c(Z)Z^{-2k})\int_0^x\mu(u)du}\mu(x)dx+\int_0^\infty R(0,Z)e^{-\lambda x-(1-c(Z)Z^{-2k})\int_0^x\eta(u)du}\eta(x)dx+$$

$$\int_0^\infty V(0,Z)e^{-\lambda x-(1-c(Z)Z^{-2k})\int_0^x V(u)du}V(x)dx-PQ_0(0)B^*(\lambda)-V_0(0)V^*(\lambda)$$

$$\mathrm{P}(0,Z)=\mathrm{P}Q(0,Z)B^*(\lambda(1-C(Z)Z^{-2k}))+R(0,Z)H^*(\lambda(1-C(Z)Z^{-2k}))$$

$$+V(0,Z)V^*(\lambda(1-C(Z)Z^{-2k}))-P\left[V_0(0)(1-V^*(\lambda))\right]-V_0(0)V^*(\lambda)$$

$$\mathrm{P}(0,Z)=\mathrm{P}Q(0,Z)B^*(\lambda(1-C(Z)Z^{-2k}))+R(0,Z)H^*(\lambda(1-C(Z)Z^{-2k}))$$

$$+V(0,Z)V^*(\lambda(1-C(Z)Z^{-2k}))-V_0(0)\,(P+qV^*(\lambda)) \qquad (5.2.54)$$

Again using (5.2.51)- (5.2.53) in (5.2.54) and solving P(0,Z) after some algebraic manipulation, we get.

$$\mathrm{Q(0,Z)} = \frac{\alpha}{Z} P(0,Z) \left[C(Z)Z^{-2k} + A^*(\lambda)(1-C(Z)^{-2k})\right] \tag{5.2.51}$$

$$\mathrm{R(0,Z)} = \bar{\alpha} P(0,Z) \left[C(Z)Z^{-2k} + A^*(\lambda)(1-C(Z)Z^{-2k})\right]$$

$$\mathrm{V(0,Z)} = \frac{q\alpha}{Z} P(0,Z) \left[C(Z)Z^{-2k} + (1-C(Z)Z^{-2k})A^*(\lambda)\right] B^*(\lambda(1-C(Z)Z^{-2k})) + V_0(0)\ (P+qV^*(\lambda)) \tag{5.2.53}$$

$$P(0,Z) = PQ(0,Z)\ B^*(\lambda(1-C(Z)Z^{-2k})) + R(0,Z)H^*(\lambda(1-C(Z)Z^{-2k})) + V(0,Z)V^*(\lambda(1-C(Z)Z^{-2k})) - V_0(0)\ (P+qV^*(\lambda)) \qquad (5.2.54)$$

$$= \frac{P\alpha}{Z} P(0,Z)\left[C(Z)Z^{-2k} + A^*(\lambda)(1-C(Z)Z^{-2k})\right] B^*(\lambda)(1-C(Z)Z^{-2k})) + \bar{\alpha}P(0,Z)\left[C(Z)Z^{-2k} + A^*(\lambda)(1-C(Z)Z^{-2k})\right]$$

$$H^*(\lambda(1-C(Z)Z^{-2k})) + \left\{ \frac{q\alpha}{Z} P(0,Z)\left[C(Z)Z^{-2k} + (1-C(Z)Z^{-2k})A^*(\lambda)\right] B^*(\lambda(1-C(Z)Z^{-2k})) + \right.$$

$$\left. V_0(0)\ (P+qV^*(\lambda)) \right\} \qquad V^*(\lambda(1-C(Z)Z^{-2k})) - V_0(0)\ (P+qV^*(\lambda))$$

$$ZP(0,Z) = P\alpha P(0,Z)\left[C(Z)Z^{-2k} + A^*(\lambda)(1-C(Z)Z^{-2k})\right] B^*(\lambda(1-C(Z)Z^{-2k})) + \bar{\alpha}ZP(0,Z)\left[C(Z)Z^{-2k} + A^*(\lambda)(1-C(Z)Z^{-2k})\right]$$

$$H^*(\lambda(1-C(Z)Z^{-2k})) + q\alpha P(0,Z)\left[C(Z)Z^{-2k} + (1-C(Z)Z^{-2k})A^*(\lambda)\right] B^*(\lambda(1-C(Z)Z^{-2k}))\ V^*(\lambda(1-C(Z)Z^{-2k})) +$$

$$V_0(0)Z\ (P+qV^*(\lambda))\ V^*(\lambda(1-C(Z)Z^{-2k})) - V_0(0)Z\ (P+qV^*(\lambda))$$

$$P(0,Z)\ \left\{\ P\alpha\left[C(Z)Z^{-2k} + A^*(\lambda)(1-C(Z)Z^{-2k})\right] B^*(\lambda(1-C(Z)Z^{-2k})) + \bar{\alpha}Z\left[C(Z)Z^{-2k} + A^*(\lambda)(1-C(Z)Z^{-2k})\right]^* \right.$$

$$\left. H^*(\lambda(1-C(Z)Z^{-2k})) + q\alpha\left[C(Z)Z^{-2k} + (1-C(Z)Z^{-2k})A^*(\lambda)\right] B^*(\lambda(1-C(Z)Z^{-2k}))\ V^*(\lambda(1-C(Z)Z^{-2k})) - Z\ \right\}$$

$$-\ V_0(0)Z\ (P+qV^*(\lambda))\left[1 - V^*(\lambda(1-C(Z)Z^{-2k})\right] = 0$$

$$P(0,Z)\ \left\{\ \left[C(Z)Z^{-2k} + A^*(\lambda)(1-C(Z)^{-2k})\right]\left[\alpha B^*(\lambda(1-C(Z)Z^{-2k}))\right]\ (P+qV^*(\lambda\ (1-C(Z)Z^{-2k})) + \bar{\alpha}Z\ H^*(\lambda(1-C(Z)Z^{-2k}))\right.$$

$$-\quad Z = V_0(0)Z\ (P+qV^*(\lambda))\left[1 - V^*(\lambda(1-C(Z)Z^{-2k}))\right]$$

$$P(0,Z)=\frac{V_0(0)Z\,(P+qV^*(\lambda))\left[1-V^*(\lambda(1-C(Z)Z^{-2k}))\right]}{\left\{\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)^{-2k})\right]\{\left[\alpha B^*(\lambda(1-C(Z)Z^{-2k}))\right](P+qV^*(\lambda(1-C(Z)Z^{-2k}))\}+\bar{\alpha}Z\,H^*(\lambda(1-C(Z)Z^{-2k}))\right\}-Z} \tag{5.2.55}$$

Finally, combining (5.2.42)-(5.2.45), (5.251)- (5.2.53) and (5.2.55), we obtain the required result (5.2.30)- (5.2.33) for the limiting probability generating functions P(X,Z), Q(X,Z), R(X,Z) and V(X,Z)

$$P(0,Z)\,e^{-\lambda x-\int_0^x r(u)du}=\frac{V_0(0)Z(P+qV^*(\lambda))\left[1-V^*(\lambda(1-C(Z)Z^{-2k}))\right]e^{-\lambda x-\int_0^x r(u)du}}{\left\{\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]\left[\alpha B^*(\lambda(1-C(Z)Z^{-2k}))\right](P+qV^*(\lambda(1-C(Z)Z^{-2k}))\}+\bar{\alpha}ZH^*(\lambda(1-C(Z)Z^{-2k}))\right\}-Z}$$

$$P(X,Z)=\frac{V_0(0)Z(P+qV^*(\lambda))\left[1-V^*(\lambda(1-C(Z)Z^{-2k}))\right]e^{-\lambda x}\left[1-A(x)\right]}{D(Z)} \tag{5.2.56}$$

Put in (5.2.51)

$$Q(0,Z)=\frac{\alpha}{Z}P(0,Z)\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]$$

$$Q(0,Z)\, e^{-\lambda x-(1-C(Z)Z^{-2k})\int_0^x \mu(u)du} = \frac{V_0(0)\left\{\alpha(P+qV^*(\lambda))\left[1-V^*(\lambda(1-C(Z)Z^{-2k}))\right]*\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]e^{-\lambda x-(1-C(Z)Z^{-2k})\int_0^x \eta(u)du}\right\}}{D(Z)}$$

$$Q(X,Z) = \frac{V_0(0)\left\{\alpha(P+qV^*(\lambda))(1-V^*(\lambda(1-C(Z)Z^{-2k})))\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]e^{-\lambda x}\left[1-B(x)\right]\right\}}{D(Z)} \qquad (5.2.57)$$

Put in (5.2.52), we get

$$R(0,Z) = \overline{\alpha}P(0,Z)\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]$$

$$R(0,Z)\, e^{-\lambda x-(1-C(Z)Z^{-2k})\int_0^x \eta(u)du} = \frac{V_0(0)\left\{\overline{\alpha}Z(P+qV^*(\lambda))\left[1-V^*(\lambda(1-C(Z)Z^{-2k}))\right]\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]e^{-\lambda x-(1-C(Z)Z^{-2k})\int_0^x \eta(u)du}\right\}}{D(Z)}$$

$$R(X,Z) = \frac{V_0(0)\left\{\overline{\alpha}Z(P+qV^*(\lambda))\left[1-V^*(\lambda(1-C(Z)Z^{-2k}))\right]\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]e^{-\lambda x}\left[1-H(x)\right]\right\}}{D(Z)} \qquad (5.2.58)$$

Putting in equation (5.2.53) we get

$$V(0,Z) = \frac{q\alpha}{Z}P(0,Z)\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]B^*(\lambda(1-C(Z)Z^{-2k}))+V_0(0)\left[P+qV^*(\lambda)\right]$$

$$= \frac{q\alpha V_0(0)(P+qV^*(\lambda)\left[1-V^*(\lambda(1-C(Z)Z^{-2k}))\right]\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]B^*(\lambda(1-C(Z)Z^{-2k}))}{\left\{\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})\right]\left\{\alpha B^*(\lambda(1-C(Z)Z^{-2k}))\left\{P+qV^*(\lambda(1-C(Z)Z^{-2k}))\right\}+\overline{\alpha}ZH^*(\lambda(1-C(Z)Z^{-2k}))\right\}-Z} + V_0(0)\left[P+qV^*(\lambda)\right]$$

$$= \frac{1}{D(Z)} \left[\begin{array}{l} V_0(0)(P+qV^*(\lambda)\{\{\alpha q[1-V^*(\lambda(1-C(Z)Z^{-2k})][C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})]B^*(\lambda(1-C(Z)Z^{-2k}))\}\} \\ +[[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})]][\alpha B^*(\lambda(1-C(Z)Z^{-2k}))\}(P+qV^*(\lambda(1-C(Z)Z^{-2k}))\}+\bar{\alpha}ZH^*(\lambda(1-C(Z)Z^{-2k})\ \}-Z\ \} \end{array} \right]$$

$$= \frac{V_0(0)(P+qV^*(\lambda)\{[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})][\alpha B^*(\lambda(1-C(Z)Z^{-2k})]\}q[1-V^*(\lambda(1-C(Z)Z^{-2k}]+(P+qV^*(\lambda(1-C(Z)Z^{-2k}))+\bar{\alpha}ZH^*(\lambda(1-C(Z)Z^{-2k}))-Z}{D(Z)}$$

$$V(0,Z) = \frac{V_0(0)(P+qV^*(\lambda)\{[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})][\alpha B^*(\lambda(1-C(Z)Z^{-2k}))]\}[q-qV^*(\lambda(1-C(Z)Z^{-2k}))+(P+qV^*(\lambda(1-C(Z)Z^{-2k}))]+\bar{\alpha}ZH^*(\lambda(1-C(Z)Z^{-2k}))-Z}{D(Z)}$$

$$V(0,Z)\ e^{-\lambda x-(1-C(z)Z^{-2k})\int_0^x V(u)du} = \{V_0(0)(P+qV^*(\lambda))\{[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})]\alpha B^*(\lambda(1-C(Z)Z^{-2k}))\}+\bar{\alpha}ZH^*(\lambda(1-C(Z)Z^{-2k}))-Z\}e^{-\lambda x-(1-C(z)Z^{-2k})\int_0^x V(u)du}$$

$$V(X,Z) = \frac{V_0(0)(P+qV^*(\lambda)\{[C(Z)Z^{-2k}+A^*(\lambda)(1-C(Z)Z^{-2k})][\alpha B^*(\lambda(1-C(Z)Z^{-2k})]\}+\bar{\alpha}ZH^*(\lambda(1-C(Z)Z^{-2k}))-Z\ \}e^{-\lambda x}[1-V(x)]}{D(Z)} \qquad (5.2.5\ 9)$$

We define the partial probability generating functions as

$P(Z) = \int_0^\infty P(x,Z)dx \qquad Q(Z) = \int_0^\infty Q(x,Z)dx$

$R(Z) = \int_0^\infty R(x,Z)dx \qquad V(Z) = \int_0^\infty V(x,Z)dx$

and the probability generating function of the number of customers in the system is K(Z) = P(Z) + ZQ(Z) + V(Z).

Here P(Z) is the probability generating function of the orbit size when the server is idle, Q(Z) is the probability generating function of the orbit size when the server is busy. R(Z) is the probability generating function of the orbit size when the server is under repair. V(Z) is the probability generating function of the orbit size when the server is on vacation and $V_0(0)$ is the probability that the server is on vacation and no server in the system. Then the main results given by the theorem.

Theorem 2 : - if $\alpha\lambda\overline{C}_1(B_1+qV_1)+\overline{\alpha}(1+\overline{C}_1\lambda D_1)+\overline{C}_1(1-A^*(\lambda))<1$ then

$$P(Z)=\frac{V_0(0)Z(P+qV^*(\lambda))(1-V^*(\lambda(1-C(Z)Z^{-2k}))(1-A^*(\lambda))}{\lambda D(Z)} \quad (5.2.60)$$

$$Q(Z)=\frac{V_0(0)\{\alpha(P+qV^*(\lambda))(1-V^*(\lambda(1-c(Z)Z^{-2K}))\}c(Z)Z^{-2K}+A^*(\lambda)(1-c(Z)Z^{-2K})\}[1-B^*(\lambda)]\}}{\lambda D(Z)} \quad (5.2.61)$$

$$R(Z)=\frac{V_0(0)\{\overline{\alpha}Z(P+qV^*(\lambda))(1-V^*(\lambda(1-c(Z)Z^{-2K}))\}c(Z)Z^{-2K}+A^*(\lambda)(1-c(Z)Z^{-2K})\}[1-H^*(\lambda)]\}}{\lambda D(Z)} \quad (5.2.62)$$

$$V(Z)=\frac{V_0(0)(P+qV^*(\lambda))\{[c(Z)Z^{-2K}+A^*(\lambda)(1-c(Z)Z^{-2K})][\alpha B^*(\lambda(1-c(Z)Z^{-2K})]+\overline{\alpha}ZH^*(\lambda(1-c(Z)Z^{-2K}))-Z\}[1-V^*(\lambda)]}{\lambda D(Z)} \quad (5.2.63)$$

Proof :-

$$\int_0^\infty P(x,Z)dx = \frac{V_0(0)Z(P+qV^*(\lambda))(1-V^*(\lambda(1-C(Z)Z^{-2k}))\int_0^\infty e^{-\lambda x}(1-A(x))dx}{D(Z)}$$

$$P(Z) = \frac{V_0(0)Z(P+qV^*(\lambda))\left[(1-V^*(\lambda(1-C(Z)Z^{-2k})\right]\left[1-A^*(\lambda)\right]}{\lambda D(Z)} \tag{5.2.64}$$

$$Q(x,Z) = \frac{V_0(0)\left\{\alpha(P+qV^*(\lambda))(1-V^*(\lambda(1-C(Z)Z^{-2k}))\left[(C(Z)Z^{-2k}+A^*(\lambda)(1-C(z)Z^{-2k})\right]e^{-\lambda x}\left[1-B(x)\right]\right\}}{D(Z)}$$

$$\int_0^\infty Q(x,Z)dx = \frac{V_0(0)\left\{\alpha(P+qV^*(\lambda))(1-V^*(\lambda(1-C(Z)Z^{-2k}))\left[(C(Z)Z^{-2k}+A^*(\lambda)(1-C(z)Z^{-2k})\right]\int_0^\infty e^{-\lambda x}\left[1-B(x)\right]dx\right\}}{D(Z)}$$

$$Q(Z) = \frac{V_0(0)\left\{\alpha(P+qV^*(\lambda))(1-V^*(\lambda(1-C(Z)Z^{-2k}))\left[(C(Z)Z^{-2k}+A^*(\lambda)(1-C(z)Z^{-2k})\right]\left[1-B^*(\lambda)\right]\right\}}{\lambda D(Z)} \tag{5.2.65}$$

Similarly

$$R(Z) = \frac{V_0(0)\left\{\bar{\alpha}Z(P+qV^*(\lambda))(1-V^*(\lambda(1-C(Z)Z^{-2k}))\left[C(Z)Z^{-2k}+A^*(\lambda)(1-C(z)Z^{-2k})\right]\left[1-H^*(\lambda)\right]\right\}}{\lambda D(Z)} \tag{5.2.66}$$

$$V(Z) = \frac{V_0(0)(P+qV^*(\lambda))\left\{\left[(C(Z)Z^{-2k}+A^*(\lambda)(1-C(z)Z^{-2k})\right]\left[\alpha B^*(\lambda(1-C(z)Z^{-2k}))\right]+\bar{\alpha}ZH^*((\lambda(1-C(z)Z^{-2k}))-Z\right\}\left[1-V^*(\lambda)\right]}{\lambda D(Z)} \tag{5.2.67}$$

Chapter - 6

Response Time Analysis to the Bulk Arrival Queue System in Communication Network

INTRODUCTION :

Queueing theory has been recognized as one of the most powerful mathematical tools for making quantative analysis of communication network. In the early 1960s it was realized that queueing theory would prove to be an effective tools for studying the throughput, response time, and other measures of performance of the network. Since that time, the literature of queueing theory and network applications have virtually exploded with analytical models for different networks. Such as for the telephone system the data transmission rates were not sufficient to meet the demand, consequently, planning for special data network began earlier in Europe then it did in United States. Since there was not yet a great demand for data services, planning was directed towards to meet the future requirement. The planners of some European nation were in many ways more ambitious then those announced by AT & T in United States, whose digital network were expected to be un-switched. The stop and wait, and continuous error detection and re-transmission schemes are widely used in such network. The researchers primary focus was the development of an appropriate model for analyzing the delays encountered in establishing a virtual circuit through a switch based LAN.

The method developed were applied to many related problems i.e. multiple link, telephone calls, receiver transmitter interaction between

switches and shared peripherals in networks. One of such approaches developed by Wilhelm, Neil C. [145] to analyze a disk subsystem in which all of the disks had the same load. Whitt, W. studied a similar method applied to markovian queueing networks as a way of extending the approximations developed by Whitt, W. Kelly, F.P. considered an approximation technique for a general class of circuit switched tele traffic models.

In 1986, Fredericks, A.A. [60] developed an approximation methods for analyzing the performance of a virtual circuit switch based LAN. The basic system consists of N input and N output parts, each with it band width divided into L equal segments for the virtual circuit from a given input to output parts, it is necessary to obtain a time, slot in each part. The author suggested a simple analytical approximation methods for the delay encountered in setting up such a circuit. Kumar, A. Singh, M.P. and Kumar V. [100] also suggested a methods to the $M/M^Y/1$ bulk service queue model for the analysis of mean response time of the communication network. Graph theory plays an important role in modeling and analyzing several problem concerns with various aspects of networking field. Also graph theoretic modeling find extensive applications in queueing theory for the networking problems. In this chapter we consider a bridge network, consists of four systems namely S_1, S_2, S_3 and S_4, which are inter connected by communication links. In this network we shall have to decided the path sets while a path in a multi-graph G consists of an alternating sequence of vertices and edges of the form v_0, e_1, v_1, e_2, v_2, ...e_{n-1}, v_{n-1}, e_n, v_n where each

edge e_i contains the vertices v_{i-1} and v_i (which appear on the sides of e_i in the sequence). The number of edges (n) is called length of the path. When there is no ambiguity. We denote a path by its sequence of vertices (v_0, v_1-------- v_n). The path is said to be closed if $v_0 = v_n$ otherwise we say that the path is from v_0 to v_n or between v_o and v_n or connects v_0 to v_n.

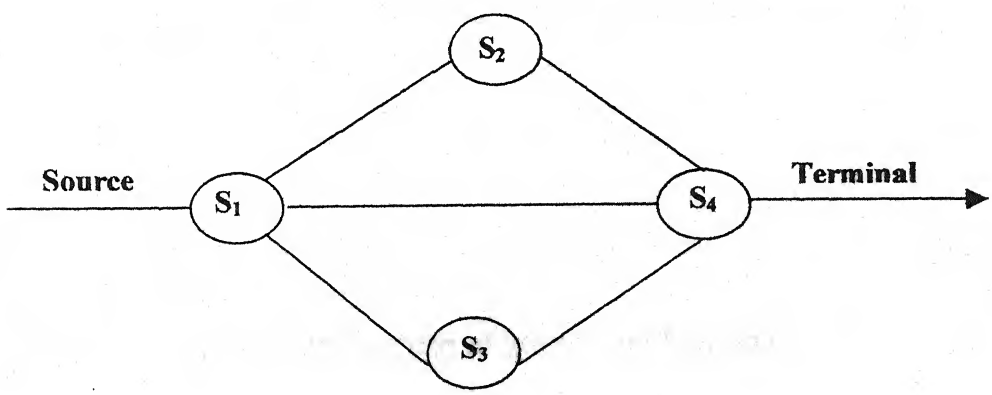
A simple path is a path in which all vertices are distinct (A path in which all edges are distinct will be called a trail). A cycle is a closed path in which all vertices are distinct except $v_o = v_n$. A cycle of length k is called a k-cycle. Here we assume that a network having "n" heterogeneous system, N = $\{S_1, S_2, \ldots\ldots\ldots S_n\}$, which are inter connected by communication links. One of the "n" system is called a source system and one of the remaining "n-1" system is labeled as terminal system. At the source system, the jobs are arriving in the form of data packets and follows the poison process. The queueing model used here in the bulk arrivals ($M^X/M/1$) type. If the source system is busy, them these jobs are store in buffer and processed latter. The job propagates over the link from source towards terminals system till they reach to terminal system. Once a job reaches to terminals system, the terminals system sends an acknowledgment of correct processing of the job to the source system the aspects of the model is the service mechanism. Each arriving data packet of jobs has variable number of modules, which arrive in bulk and is then served by the system of the network. In this model, we have obtained the mean response time. For mean response time analysis we have listed the path sets of the

networks. The tasks follow any one path from processing for source to terminals system depending upon the allocation of path by the source system and it is assumed that arrivals of tasks occurs to as an ordinary poison process with mean λ, and then they are served on the basis of first come first serve discipline. Each job of data packets has variable number of modules and their arrivals X at a time except when less then X is in the data packet. The amount of time the required for the service of any job is an exponentially distributed random variable.

In this chapter, a model for mean response time is analyzed, which is based on graph and queueing theories for making quantitative analysis in networking. Then the all the path sets of network have been listed and the $M^x/M/1$ bulk arrival queueing model is then applied for determining the mean response time of each path set.

6.1 THE PROPOSED METHODS :

Let us consider a bridge, network, consist of four system namely, S_1, S_2, S_3 and S_4, which are inter connected by communication.



The system S_1 is defined as source system, while the system S_3 is labeled as terminal system. The job enter from the source system and get released from the terminal system after getting processed. The path sets of the network are listed below, along with then number of system in a path set;

PS – 1	$S_1 - S_3$	number of system (N) = 2
PS – 2	$S_1 - S_2 - S_3$	number of system (N) = 3
PS – 3	$S_1 - S_4 - S_3$	number of system (N) = 3

6.3 MEAN WAITING TIME :

The Kol mogrov equation are given below :-

$$P^1{}_n(t) = -(\lambda + \mu) P_n(t) + \mu P_{n-1}(t) + \lambda \sum_{K=1}^{n} P_{n-K}(t) C_k \quad (n \geq 1)$$

(6.3.1)

$$P^1{}_n(t) = -\lambda P_0(t) + \mu P_1(t)$$

(6.3.2)

As $t \to \infty$ then the steady state differential difference equations are :-

$$0 = -(\lambda + \mu) P_n + \mu P_{n-1} + \lambda \sum_{K=1}^{n} P_{n-K} C_k \quad (n \geq 1) \qquad (6.3.3)$$

$$0 = -\lambda P_0 + \mu P_1 \qquad (6.3.4)$$

Solving the equation with the help of generating function

$$P(z) = \sum_{n=0}^{\infty} P_n z^n \qquad (|z| \leq 1) \qquad (6.3.5)$$

$$C(z) = \sum_{n=0}^{\infty} C_n z^n = \sum_{n=1}^{\infty} C_n z^n (|z| \le 1) \quad (6.3.6)$$

Then using from equation (6.3.3)

$$0 = -\lambda \sum_{n=0}^{\infty} P_n z^n - \mu \sum_{n=1}^{\infty} P_n z^n + \frac{\mu}{z} \sum_{n=0}^{\infty} P_{n-1} z^{n-1} + \lambda \sum_{n=1}^{\infty} \sum_{k=1}^{n} P_{n-k} C_k z^n \quad (6.3.7)$$

Here we see that $\sum_{k=1}^{n} P_{n-k} C_k$ is the probability function for the sum of the steady state system size and batch size $\sum_{n=1}^{\infty} \sum_{k=1}^{n} P_{n-k} C_k z^n = \sum_{k=1}^{n} C_k z^k \sum_{n=1}^{\infty} P_{n-k} z^{n-k} = C(z) P(z)$

Then equation (6.3.7) we get

$$0 = -\lambda P(z) - \mu [P(z) - P_0] + \frac{\mu}{z} [P(z) - P_0] + \lambda C(z) P(z)$$

$$P(z) [\lambda + \mu - \mu - \lambda . C(z)] = \mu P_0 - \frac{\mu}{z} P_0$$

$$P(z) \left[\frac{(\lambda + \mu)z - \mu - \lambda . z C(z)}{z}\right] = \mu P_0 (1 - \frac{1}{z})$$

$$P(z) = \left[\frac{\mu . P_0 (z-1)}{(\lambda + \mu)z - \mu - \lambda . z C(z)}\right]$$

$$P(z) = \left[\frac{\mu . P_0 (z-1)}{\mu(z-1) - \lambda . z(C(z) - 1)}\right]$$

$$P(z) = \left[\frac{-\mu . P_0 (z-1)}{\mu(1-z) - \lambda . z(1 - C(z))}\right] \quad (|z| \ge 1) \quad (6.3.8)$$

and using the condition P (1) = 1 we get

$$1 = \text{Lim}_{z\to 1} \text{P(z)} = \text{Lim}_{z\to 1} \left\{\frac{\mu P_0(1-z)}{\mu(1-z)-\lambda z(1-C(z))}\right\}$$

Since numerator and denominator are both zero. So we use L' Hospital rule

$$\text{X} = \text{Lim}_{z\to 1} \text{P(z)} = \text{Lim}_{z\to 1}\left\{\frac{-\mu P_0}{-\mu-\lambda+\lambda C(z)+\lambda z.dC(z)}\right\}$$

$$= \frac{-\mu P_0}{-\mu+\lambda E(X = batch\ .size\ )}$$

$$1 = \frac{-\mu P_0}{-\mu+\lambda E(X)}$$

$$-\mu P_0 = -\mu + \lambda \text{ E (X)}$$

$$P_0 = \frac{-\mu+\lambda E(X)}{-\mu}$$

$$P_0 = 1 - \frac{\lambda E(X)}{\mu}$$

$$P_0 = 1-\rho$$

Where $\rho = \dfrac{\lambda E(X)}{\mu}$ $(\rho < 1)$ (6.3.9)

Now we consider the batch size X is geometrically distributed i.e.

$$C_x = (1-\alpha)\, \alpha^{x-1} \qquad (0 \le \alpha < 1) \qquad (6.3.10)$$

Then using (6.3.4) we have

$$C\ (z) = (1-\alpha) \sum_{n=1}^{\infty} \alpha^{n-1} z^n, \qquad (|z| \le 1) \qquad (6.3.11)$$

$$C\ (z) = (1-\alpha) \frac{z}{1-\alpha z}$$

$$C(z) = \frac{z(1-\alpha)}{1-\alpha z} \qquad (|\alpha.Z| \le 1) \qquad (6.3.12)$$

From equation (6.3.8)

$$P(z) = \frac{\mu.P_0(1-z)}{\mu(1-z) - \lambda.z(1-C(z))}$$

$$P(z) = \frac{\mu.P_0(1-z)}{\mu(1-z) - \lambda.z\left[1 - \frac{z(1-\alpha)}{1-\alpha.z}\right]} = \frac{\mu(1-\rho)(1-z)}{\mu(1-z) - \lambda.z\left[1 - \frac{z(1-\alpha)}{1-\alpha.z}\right]}$$

$$= \frac{\mu(1-\rho)(1-\alpha z)(1-z)}{\mu(1-z)(1-\alpha.z) - \lambda.z\left[1 - \frac{z(1-\alpha)}{1-\alpha.z}\right]} = \frac{\mu(1-\rho)(1-\alpha z)(1-z)}{\mu(1-z) - \lambda.z\left[\frac{1-z}{1-\alpha.z}\right]}$$

$$= \frac{\mu(1-\rho)(1-\alpha z)}{\mu(1-\alpha.z) - \lambda.z} = \frac{(1-\rho)(1-\alpha z)}{1-[\alpha+(1-\alpha)\rho]z}$$

$$P(z) = (1-\rho)\left\{\frac{1}{1-[\alpha+(1-\alpha)\rho]z} - \frac{\alpha z}{1-[\alpha+(1-\alpha)\rho]z}\right\}$$

$$P(z) = (1-\rho)\left\{\sum_{n=0}^{\infty}[(\alpha+(1-\alpha)\rho)z]^n - \sum_{n=0}^{\infty}[\alpha(\alpha+(1-\alpha)\rho)]^n z^{n+1}\right\}$$

So that,

$$P_n = (1-\rho)\left\{[\alpha+(1-\alpha)\rho]^n - \alpha[\alpha+(1-\alpha)\rho]^{n-1}\right\}$$

$$P_n = (1-\rho)[\alpha+(1-\alpha)\rho]^{n-1}(1-\alpha)\rho \qquad (n\ge 0) \qquad (6.3.13)$$

6.4 LENGTH QUEUE :

The line length of queue (Ls) of the system is given by

$$Ls = \sum_{n=1}^{\infty} nP_n \qquad (6.4.1)$$

$$= \sum_{n=1}^{\infty} n.(1-\rho)[\alpha+(1-\alpha)\rho]^{n-1}(1-\alpha)\rho$$

$$= (1-\rho)(1-\alpha)\sum_{n=1}^{\infty} n.[\alpha + (1-\alpha)\rho]^{n-1}$$

$$= (1-\rho)(1-\alpha)\rho\left[\frac{1}{(1-\alpha(1-\alpha)\rho)^2}\right] \quad (6.4.2)$$

the waiting time of data packet of jobs in system for each path is as

$$W_s(i, j) = \frac{1}{\lambda}.\sum_{J=1}^{N-1} L_s(i, j)\ i = 1, 2, \ldots PS \quad (6.4.3)$$

Therefore Mean waiting time of data packet of the job, is defined as

$$T_W(i) = \sum_{J=1}^{N-1} Ws(i, j),\ i = 1, 2 \ldots PS \quad (6.4.4)$$

6.5 MEAN SERVICE TIME :

Service of the system, which bulkily transmits the job from one system to another, depends upon a batch of size K, which is exponentially distributed with the mean μ_s (i, j). Therefore the mean service time for each path, out going from the transmitting systems given as :

$$T_S(i) = \sum_{j=1}^{N-1}\left(\frac{1}{\mu_s(i, j)}\right),\ i = 1, \ldots PS \quad (6.5.1)$$

6.6 MEAN RECEIVING TIME :

Similarly, for the service of the system, whose receives the job, it is assumed that the receiving time of this system with the mean, μ_R (i, j) obeys exponential law and is as,

$$T_R(i) = \sum_{j=1}^{N-1}\left(\frac{1}{\mu_R(i, j)}\right),\ i = 1, \ldots PS \quad (6.6.1)$$

6.7 MEAN RESPONSE TIME :

Mean Response Time (T_{MRT}) is defined as the sum of the mean Waiting Time and Mean Service Time of processors, which transmit, plus the Mean Receiving Time of the processor, which receives, which receives the jobs i.e.

$$T_{MRT}(i) = [T_W(i) + T_S(i) + T_R(i)] \qquad (6.7.1)$$

Where T_W (i), T_S (i) and T_R (i) are given in the equation (6.4.4.), (6.4.1), (6.5.1), respectively

6.8 Computation of Numerical Results :

For the computation of numerical results, Mean Response Time has been obtained for different value s of x i.e. different data packet size, which are shown through table 1 to 4. Let us assume the service rate for S_1, S_2, S_3, S_4 are 1.5, 2.0, 2.5 and 2.5 respectively and receiving rate for S_1, S_2, S_3, S_4 are 0.5, 1.0, 1.5 and 2.0 respectively. For the various path sets of the network the different values of batch size x and λ are1,2,3 E (x) are 0.5, 0.7, 0.8, and 0.9, 0.5, 0.7 for α. The Mean Waiting Time for each has been obtained. It is given in the following table-1.

T_W (i)	Mean Waiting Time		
	X = 1	X = 2	X = 3
T_W (1)	0.0234	0.0771	0.0231
T_W (2)	0.0182	0.0358	0.0086
T_W (3)	0.0065	0.0061	0.0050

The Mean Service Time for each have been obtained, which is given in the following table – 2 :

Table – 2

T_S (i)	Mean Service Time
T_S (1)	1.0666
T_S (2)	1.5666
T_S (3)	1.4666

The Mean Receiving Time for each have been calculated. It is given in the following table –3:

Table – 3

T_R (i)	Mean Receiving Time
T_R (1)	2.6666
T_R (2)	3.6666
T_R (3)	3.1666

Finally the Mean Response Time, for the different values of batch size x and λ i.e., 1,2,3 for each path have been obtained. It is shown in the following table-4

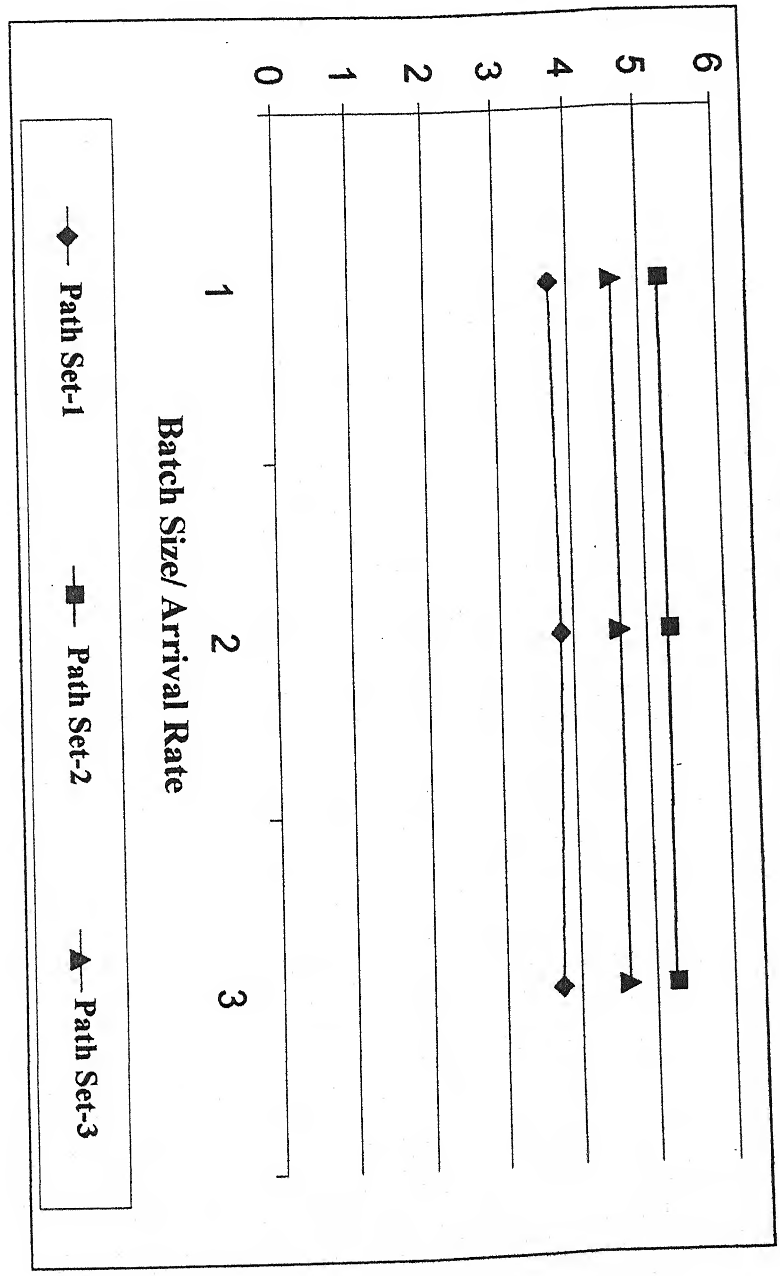
Table – 4

T_{MRT} (i)	Mean Response Time		
$T_{MRT}(1)$	3.7566	3.8103	3.7563
$T_{MRT}(2)$	5.2514	5.2690	5.2418
$T_{MRT}(3)$	4.6397	4.6393	4.6382

6.9 DISCUSSION :

The mean response time $M^x/M/1$ bulk arrival queue model to the various path sets for the bridge network as shown in figure-1, has been obtained for the different value of batch size and arrival rate i.e. 1,2,3, respectively are shown in Table-4. Graphical representation of the batch size/arrivals rate vs mean response time has been shown in graph-1. It can be stated, that the mean response time is lesser for the path set-1. Further for the path set-2, the mean response time is higher as the service rate is poor. In the case the path set-3, it is between the path set-1 and path set-2, shows that the mean response time decrease with the increase service rate. The model discuss in this chapter, would be useful to the network system designer working in the field of networking. The decision of routing strategy in the design of network may also be facilitated by the present.

Mean Response Time



Graph-1
Batch Size/Arrival Rate
vs
Mean Response Time

Reference

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